Engineering Fluid Mechanics
Lecture X

Chapter 5
Dimensional analysis

Review-Dimensions and Units

- A ***dimension*** is the measure by which a physical variable is expressed quantitatively.
- A ***unit*** is a particular way of attaching a number to the quantitative dimension.
- For example, length is a dimension associated to variables, such as, distance, displacement, width, and height.
- The unit of length can be m, cm, or mm.

Review-Dimensions and Units

- Primary dimensions
  - In fluid mechanics there are only four ***primary dimensions*** from which all other dimensions can be derived: ***mass (M), length (L), time (T) and temperature (Θ)***.

Primary dimension	SI unit	BG unit	Conversion factor
Mass $\{M\}$	Kilogram (kg)	Slug	1 slug = 14.5939 kg
Length $\{L\}$	Meter (m)	Foot (ft)	1 ft = 0.3048 m
Time $\{T\}$	Second (s)	Second (s)	1 s = 1 s
Temperature $\{\Theta\}$	Kelvin (K)	Rankine (°R)	1 K = 1.8°R

Review-Dimensions and Units

- Secondary dimensions
  - Dimension of *velocity* and *acceleration* are {LT^{-1}} and {LT^{-2}}, respectively.
  - Question: what is the dimension of ***force***?

Secondary dimension	SI unit	BG unit	Conversion factor
Area {L^2}	m^2	ft^2	$1\ m^2 = 10.764\ ft^2$
Volume {L^3}	m^3	ft^3	$1\ m^3 = 35.315\ ft^3$
Velocity {LT^{-1}}	m/s	ft/s	1 ft/s = 0.3048 m/s
Acceleration {LT^{-2}}	m/s^2	ft/s^2	$1\ ft/s^2 = 0.3048\ m/s^2$
Pressure or stress {$ML^{-1}T^{-2}$}	$Pa = N/m^2$	lbf/ft^2	$1\ lbf/ft^2 = 47.88\ Pa$
Angular velocity {T^{-1}}	s^{-1}	s^{-1}	$1\ s^{-1} = 1\ s^{-1}$
Energy, heat, work {ML^2T^{-2}}	$J = N \cdot m$	$ft \cdot lbf$	$1\ ft \cdot lbf = 1.3558\ J$
Power {ML^2T^{-3}}	W = J/s	$ft \cdot lbf/s$	$1\ ft \cdot lbf/s = 1.3558\ W$
Density {ML^{-3}}	kg/m^3	$slugs/ft^3$	$1\ slug/ft^3 = 515.4\ kg/m^3$
Viscosity {$ML^{-1}T^{-1}$}	$kg/(m \cdot s)$	$slugs/(ft \cdot s)$	$1\ slug/(ft \cdot s) = 47.88\ kg/(m \cdot s)$
Specific heat {$L^2T^{-2}\Theta^{-1}$}	$m^2/(s^2 \cdot K)$	$ft^2/(s^2 \cdot °R)$	$1\ m^2/(s^2 \cdot K) = 5.980\ ft^2/(s^2 \cdot °R)$

Review-Dimensions and Units

- Dimensionally homogeneous condition
  - All theoretical equations in mechanics (and in other physical sciences) are dimensionally homogeneous; i.e., each term in the equation has the same dimension.
  - An example is the equation from physics for a body falling with negligible air resistance:

$$S = S_0 + V_0 t + \frac{1}{2} g t^2$$

  - Each term in this relation has dimensions of length {L}. The factor 1/2, is a pure (dimensionless) number.

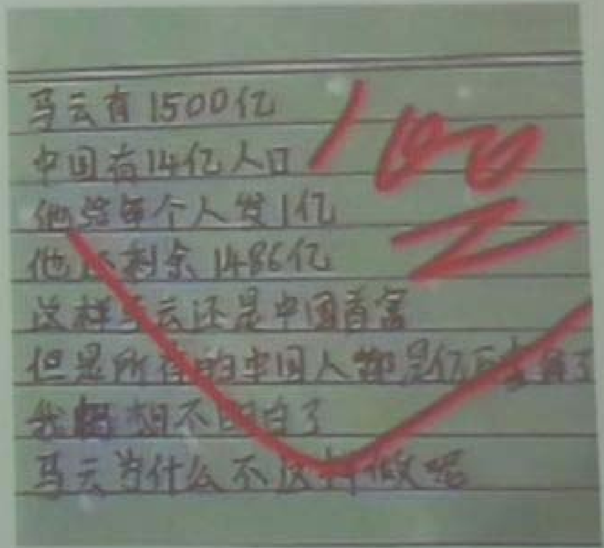
1分=0.01元=0.1元X 0.1元=1角X 1角=10分X 10分=100分=1元.

对吗?

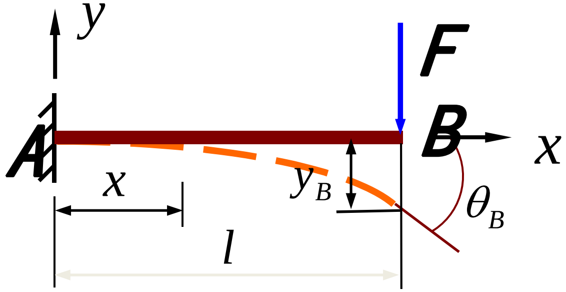
物理量纲的幂次定律: 任何物理量的量纲公式都是基本量纲的幂次单项式的形式

$$Y = L^a m^b t^c,\ 其中a, b, c为实数$$

量纲一致性定律: 每个在公式中相加的量其量纲必须一致



例1　求梁的转角方程和挠度方程，并求最大转角和最大挠度，梁的*EI*已知。



解　1）由梁的整体平衡分析可得：

$$F_{Ax}=0,\ F_{Ay}=F(\uparrow),\ M_A=Fl(\circlearrowleft)$$

2）写出*x*截面的弯矩方程

$$M(x)=-F(l-x)=F(x-l)$$

3）列挠曲线近似微分方程并积分

$$EI\frac{d^2y}{dx^2}=M(x)=F(x-l)$$

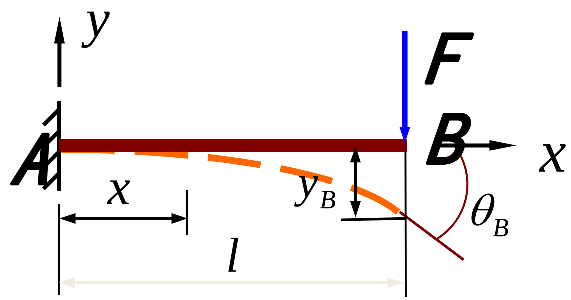
积分一次

$$EI\frac{dy}{dx}=EI\theta=\frac{1}{2}F(x-l)^2+C$$

再积分一次

$$EIy=\frac{1}{6}F(x-l)^3+Cx+D$$

4）由位移边界条件确定积分常数



$$\begin{cases} x=0, & \theta_A=0 \\ x=0, & y_A=0 \end{cases}$$

代入求解

$$C=-\frac{1}{2}Fl^2, \quad D=\frac{1}{6}Fl^3$$

5）确定转角方程和挠度方程

$$EI\theta=\frac{1}{2}F(x-l)^2-\frac{1}{2}Fl^2$$

$$EIy=\frac{1}{6}F(x-l)^3-\frac{1}{2}Fl^2x+\frac{1}{6}Fl^3$$

注意检查量纲！

6）确定最大转角和最大挠度

$$x=l, \quad \theta_{max}=|\theta_B|=\frac{Fl^2}{2EI}, \quad y_{max}=|y_B|=\frac{Fl^3}{3EI}$$

量纲分析法主要作用：

1.确定物理量之间的关系（写公式）

2.减少实验中变量的个数（**PI**定理），建立缩减模型实验和实际物体物理量之间的关系。

3.减少支配物理现象的方程中参数的个数。

量纲分析应用1：写公式

对所设问题有一定了解，在实验和经验的基础上利用量纲齐次原则来确定各物理量之间的关系.

例 单摆运动

将质量为m 的一个小球系在长度为l 的线的一端, 稍偏离平衡位置后小球在重力mg的作用下，做往复摆动. 忽略阻力, 求摆动周期t的表达式.

求解 考虑问题中出现的物理量t、m、l、g，假设它们之间有关系式

$$t = \lambda m^{\alpha_1} l^{\alpha_2} g^{\alpha_3}$$

其中α_1, α_2, α_3是待定常数, λ是无量纲的比例常数.

上式的量纲表达式为

$$[t] = [m]^{\alpha_1} [l]^{\alpha_2} [g]^{\alpha_3}$$

将[t]=T, [m]=M, [l]=L, [g]=LT^{-2} 代入得

$$T = M^{\alpha_1} L^{\alpha_2+\alpha_3} T^{-2\alpha_3}$$

按照量纲齐次性, 有

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ -2\alpha_3 = 1 \end{cases}$$

求解为

$$\alpha_1 = 0, \alpha_2 = \frac{1}{2}, \alpha_3 = -\frac{1}{2}$$

得

$$t = \lambda\sqrt{\frac{l}{g}}$$

缺点:

（1）必须知道影响问题的所有物理量

（2）只能推导多项式形式的公式。

5.1 Introduction

- **The dimensional analysis**
  - Dimensional analysis is a method for reducing the number of variables which affect a given physical phenomenon into dimensionless parameters.
  - If a phenomenon depends upon *n* dimensional variables, dimensional analysis will reduce the problem to only *k* dimensionless variables.

5.1 Introduction

- **The dimensional analysis**
  - One benefit of the dimensional analysis is enormous savings in time and money.
  - Suppose one knew that the force F on a particular body immersed in a stream of fluid depended only on the body length L, stream velocity V, fluid density ρ, and fluid viscosity μ, that is,

$$F = f\left(L, V, \rho, \mu\right)$$

5.1 Introduction

- **The dimensional analysis**
  - Generally speaking, it takes about 10 experimental points to define a curve.
  - To find the effect of body length, we have to run the experiment for 10 lengths L.
  - For each L we need 10 values of V, 10 values of ρ, and 10 values of μ, making a grand total of 10^4 experiments.

What a great cost! Could we reduce it?

5.1 Introduction

- **The dimensional analysis**
  - However, with dimensional analysis, we can immediately reduce the above equation to the equivalent form

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \qquad C_F = g\left(Re\right)$$

  - The function *g* is different mathematically from the original function *f*, but it contains all the same information.
  - Nothing is lost in a dimensional analysis. And think of the savings: We can establish g by running the experiment for only 10 values of the single variable called the *Reynolds number*.

5.2 The PI theorem

- **The PI theorem**
  - The scheme given in 1914 by Buckingham for reducing a number of dimensional variables into a smaller number of dimensionless groups is now called the Buckingham pi theorem.
  - The name pi comes from the mathematical notation Π, meaning a product of variables. The dimensionless groups found from the theorem are power products denoted by Π_1, Π_2, Π_3, etc.

5.2 The PI theorem

- **The PI theorem**
  - The first part of the pi theorem explains what reduction in variables to expect:
  - If a physical process satisfies the dimensionally homogeneous condition and involves ***n*** dimensional variables, it can be reduced to a relation between only ***k*** dimensionless variables or Π's. The reduction ***j=n-k*** =(the maximum number of variables which do not form a pi among themselves) and is always ≤ (the number of dimensions describing the variables).

5.2 The PI theorem

- **The PI theorem**
  - The second part of the theorem shows how to find the pis one at a time:
  - Find the reduction *j*, then select *j* scaling variables which do not form a *pi* among themselves. Each desired pi group will be a power product of these *j* variables plus one additional variable which is assigned any convenient non-zero exponent. Each *pi* group thus found is independent.

5.2 The PI theorem

- **The PI theorem**
  - To be specific, suppose that the process involves five variables

$$v_1 = f\left(v_2, v_3, v_4, v_5\right)$$

  - Suppose that there are three dimensions {MLT} and we search around and find that indeed j=3. Then k=5-3=2 and we expect, from the theorem, two and only two pi groups.

5.2 The PI theorem

- **The PI theorem**
  - Pick out three convenient variables which do not form a pi, and suppose these turn out to be v_2, v_3, and v_4.
  - Then the two pi groups are formed by power products of these three plus one additional variable, either v_1 or v_5:

$$\Pi_1 = v_2^{a_1} v_3^{b_1} v_4^{c_1} v_1 = M^0 L^0 T^0$$

$$\Pi_2 = v_2^{a_2} v_3^{b_2} v_4^{c_2} v_5 = M^0 L^0 T^0$$

  - Question: how to determine the power coefficients?

5.2 The PI theorem

- **The PI theorem**
  - Typically, six steps are involved:
  - 1. List and count the *n* variables involved in the problem.
  - 2. List the dimensions of each variable according to {MLTΘ}.
  - 3. Find j. Initially guess j equal to the number of different dimensions present. If no luck, reduce j by 1 and look again.
  - 4. Select j parameters which do not form a pi product.
  - 5. Add one additional variable to j repeating variables, and form a power product. Algebraically find the exponents which make the product dimensionless.
  - 6. Write the final dimensionless function, and check your work to make sure all pi groups are dimensionless.

5.2 The PI theorem

- **Example 1**
  - The fluid force F on a particular body immersed in a stream of fluid depended only on the body length L, stream velocity U, fluid density ρ, and fluid viscosity μ.
  - Find the dimensionless parameters of the problem according to the pi theorem.

5.2 The PI theorem

- **Solution**

1.Write the function and count variables:

$$F = f\left(L, U, \rho, \mu\right)$$

there are five variables (n =5)

2. List dimensions of each variable.

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

5.2 The PI theorem

3.Find j. No variable contains the dimension Θ, and so j is less than or equal to 3 (MLT). We inspect the list and see that L, U, and ρ cannot form a pi group because only ρ contains mass and only U contains time. Therefore j does equal 3, and n - j = 5 - 3 = 2 = k. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

Select repeating j variables. The group L, U, ρ we found in will do fine.

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

5.2 The PI theorem

上一步事实上是确定以下矩阵的秩，并在列向量中选出一个极大线性无关组：

$$
\begin{array}{c} \\ M \\ L \\ T \end{array}
\begin{array}{c}
\begin{array}{ccccc} F & L & U & \rho & \mu \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -3 & -1 \\ -2 & 0 & -1 & 0 & -1 \end{bmatrix}
\end{array}
$$

5.2 The PI theorem

4. Combine *L*, *U*, ρ with one additional variable, in sequence, to find the two pi products.

First add force to find Π_1. You may select any exponent on this additional term as you please, to place it in the numerator （分子） or denominator （分母） to any power. Since F is the output, or dependent, variable, we select it to appear to the first power in the numerator:

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (\ MLT^{-2}) = M^0 L^0 T^0$$

5.2 The PI theorem

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

⇩

$$a + b - 3c + 1 = 0$$

$$c + 1 = 0$$

$$-b \qquad -2 = 0$$

⇨ $a = -2 \qquad b = -2 \qquad c = -1$

$$\Pi_1 = L^{-2} U^{-2} \rho^{-1} F = \frac{F}{\rho U^2 L^2} = C_F$$

Ans.

5.2 The PI theorem

Finally, add viscosity to L, U, and ρ to find Π_2. Select any power you like for viscosity. By hindsight and custom, we select the power -1 to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

$$\Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \frac{\rho U L}{\mu} = \mathrm{Re}$$ Ans.

The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right)$$

5.2 The PI theorem

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right)$$

$$C_F = g(\mathrm{Re})$$

$$\text{If} \quad \mathrm{Re}_m = \mathrm{Re}_p \quad \text{then} \quad C_{Fm} = C_{Fp}$$

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

5.2 The PI theorem

- **Example 2**

  Assume that the tip deflection δ of a cantilever beam is a function of the tip load P, beam length L, area moment of inertia I, and material modulus of elasticity E; that is, $\delta = f(P, L, I, E)$. Rewrite this function in dimensionless form.

5.2 The PI theorem

- **Solution**

List dimensions of each variable.

δ	P	L	I	E
$\{L\}$	$\{MLT^{-2}\}$	$\{L\}$	$\{L^4\}$	$\{ML^{-1}T^{-2}\}$

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} \delta & P & L & I & E \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 4 & -1 \\ 0 & -2 & 0 & 0 & -2 \end{pmatrix} \end{array}$$

⇨ j=2

5.2 The PI theorem

- **Solution**

With j = 2, we select *L* and *E* as two variables which cannot form a pi group and then add other variables to form the three desired pis:

$$\Pi_1 = L^a E^b I^1 = (L)^a (ML^{-1}T^{-2})^b (L^4) = M^0 L^0 T^0$$

$$\Pi_2 = L^a E^b \mathrm{P}^1 = (L)^a (ML^{-1}T^{-2})^b (MLT^{-2}) = M^0 L^0 T^0$$

$$\Pi_3 = L^a E^b \delta^1 = (L)^a (ML^{-1}T^{-2})^b (L) = M^0 L^0 T^0$$

方程看似不够，但确实可以解

$$\frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right)$$

Ans.

Homework

- **Problems P5.23, P5.25， P5.26,P5.47 in Chapter 5 of the book.**

- 注：**P5.47**下次课讲完再做。**5-30**交作业