文章编号:2095-6134(2018)05-0577-05

Constructing integrating factor to distinguish between the center and the focus*

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(Received 2 May 2017; Revised 22 September 2017)

Zhi J H, Chen Y F. Constructing integrating factor to distinguish between the center and the focus[J]. Journal of University of Chinese Academy of Sciences, 2018, 35(5):577-581.

Abstract In this paper we consider real differential systems which at singular points have purely imaginary eigenvalues. We show that the real invariant algebraic curves of the system are not identically zero in some deleted neighborhood $U^{\circ}(O)$. A method for seeking local first integrals or integrating factors is proposed, which can be used in determining the types of the equilibrium points.

Keywords invariant algebraic curve; first integral; integrating factor; equilibrium point

CLC number: O29; O35 **Document code**: A **doi**: 10.7523/j.issn.2095-6134.2018.05.001

基于构造积分因子的中心和焦点判别方法

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摘 要 研究具有一对纯虚特征值的实系统的实不变代数曲线在原点空心邻域非零的性质,使用不变代数曲线和指数因子构造局部首次积分或积分因子,提出进行平衡点类型判别的方法。

关键词 不变代数曲线; 首次积分; 积分因子; 平衡点

Let k denote an effective field of characteristic zero. We consider planar vector fields and their associated differential equations that have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y), \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y), \tag{1}$$

where P and Q are polynomials with the coefficients in the real field. If the linearization of system (1) has purely imaginary eigenvalues, then, by a linear transformation, the system can be converted to the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + P_2 + \cdots + P_m,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + Q_2 + \cdots + Q_m. \tag{2}$$

Distinguishing between a center and a focus is one challenge of nonlinear differential equations with long history.

In this paper we present a method to improve the methods of distinguishing between a center and a focus.

* Supported by the National Natural Science Foundation of China(11271363)

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1 Study on the problem of center and focus

The theorem of Poincare-Liapunov says that the origin of polynomial system (2) is a center if and only if the system has a non-constant analytic first integral in a neighborhood. In searching for sufficient conditions for a center, both Poincare and Liapunov's works involve the idea of trying to find an analytic function $F(x,y)$ in a neighborhood of $O(0,0)$, where $F(x,y) = \sum_{i=2}^{\infty} F_i(x,y)$ with $F_2 = x^2 + y^2$. Based on Poincare's method, Wang[1] described a mechanical procedure for constructing the Liapunov constants and Liapunov functions of autonomous differential system with the center and focus type. Zhi and Chen[2] gave a method to identify the type of critical point by constructing Dulac-Cherkas function.

Darboux[3] showed how to construct the first integrals of planar polynomial vector fields possessing sufficient invariant algebraic curves. In particular, he proved that the planar vector field of degree m with at least $\frac{m(m+1)}{2}+1$ invariant algebraic curves has a first integral. The ideas pioneered by Darboux have been developed significantly in recent years by Prelle and Singer.

It is known that the origin is a center if system (2) has an analytic integrating factor of the form $\mu(x,y) = 1 + \sum_{k=1}^{\infty} \mu_k(x,y)$. Prelle and Singer[4] demonstrated in 1983 that if a polynomial system has an elementary first integral, then the system admits an integrating factor R such that R^n is in $\mathbb{C}(x,y)$. Singer[5] and Christopher[6] showed that if a polynomial system has a Liouvillian first integral, then the system has an integrating factor of the form $R = \exp(g/h)\prod f_i^{\lambda_i}$. Christopher[7], Pearson et al.[8] and Pereira[9] used invariant curves that do not vanish at the origin in seeking local integrals or integrating factors. They demonstrated how computer algebra can be effectively employed in the search for necessary and sufficient conditions for critical points of such systems to be centers. The systematic method is used in cubic systems which was previously intractable.

2 Invariant algebraic curve

Let $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be the vector field associated with the planar polynomial differential system (1) and denote $m = \max\{\deg(P), \deg(Q)\}$ the degree of the vector field. An algebraic curve $f=0$, with $f \in k[x,y]$, is an invariant algebraic curve for the vector field D, if there is a polynomial L_f with the degree of at most $(m-1)$ such that

$$D(f) = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = fL_f. \tag{3}$$

The polynomial L_f is called the cofactor of f. Considering differential systems (2), we can obtain the special proposition of real invariant algebraic curve.

Theorem 2.1 Suppose real coefficient polynomial $f = \sum_{k=s}^{r} f_k = 0$, where $f_s \neq 0$, is an invariant algebraic curve of (2) and L_f is the cofactor of f, then $L_f(0,0) = 0$ and $f_s = (x^2+y^2)^p$ with $s = 2p \geqslant 0$.

Proof Let $f_s = \sum_{i=0}^{s} c_{i,s-i}x^iy^{s-i}$ and $L_f(0,0) = -\lambda$. The lowest order terms in equation (3) give

$$y\frac{\partial f_s}{\partial x} - x\frac{\partial f_s}{\partial y} = -\lambda f_s, \tag{4}$$

so that

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ -s & \lambda & 2 & 0 & 0 \\ 0 & -s+1 & \lambda & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & s \\ 0 & 0 & 0 & -1 & \lambda \end{pmatrix} \begin{pmatrix} c_{0,s} \\ c_{1,s-1} \\ \vdots \\ c_{s-1,1} \\ c_{s,0} \end{pmatrix} = 0. \tag{5}$$

No matter $\lambda > 0$ or $\lambda < 0$, the coefficient matrix of equation (5) is through the primary transformation into an upper triangular matrix or a lower triangular matrix, whose diagonal elements are non zero. Then equation (5) has the unique zero solution, which leads to a contradiction. Hence $L_f(0,0) = \lambda = 0$. For equation (5), if i and j are odd numbers, $c_{i,s-i} = c_{s-j,j} = 0$. From $f_s \neq 0$, s is an even number. Let $D_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. Since $D_1((x^2 + y^2)h(x,y)) = (x^2+y^2)D_1(h(x,y))$ and the rank of coefficient matrix $R \geqslant s$, then $f_s = (x^2+y^2)^p$ with $s = 2p \geqslant 0$.

An algebraic curve $f = 0$ is said to be irreducible if it has only one component, and reduced if all components appear with multiplicity one. A reducible polynomial is split into its irreducible factors which also define invariant curves of system. Proposition 2.1 can be found in Ref. [10].

Proposition 2.1 $f \in k[x,y]$ and $n = \deg(f)$. Let $f = f_1^{n_1}f_2^{n_2}\cdots f_r^{n_r}$ be its factorization in irreducible factors. Then, for a vector field D, $f = 0$ is an invariant algebraic curve with the cofactor L_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1,2,\cdots,r$ with the cofactor L_{f_i} and $L_f = n_1L_{f_1} + n_2L_{f_2} + \cdots + n_rL_{f_r}$.

In order to obtain more invariant algebraic curves, we can compute complex algebraic curves. Let $f(x,y) \in \mathbb{C}[x,y]$ and $f(x,y) = 0$ is an invariant algebraic curve with cofactor $L_f(x,y)$. Since system (2) has the real coefficients, then theconjugate $\bar{f}(x,y)$ is also an invariant algebraic curve with cofactor $\overline{L_f}(x,y)$. Let $\mathrm{Re}f(x,y)$ be the real part of the polynomial $f(x,y)$ and $\mathrm{Im}f(x,y)$ be its imaginary part. The result on complex algebraic curve is given as follows.

Theorem 2.2 Suppose $f(x,y) \in \mathbb{C}[x,y]$ is an invariant algebraic curve of system (2) with cofactor $L_f(x,y)$, then $\mathrm{Re}L_f(x,y)$ vanish at the origin. If $L_f(0,0) = 0$, the lowest order terms of $\mathrm{Re}f(x,y) = \sum_{j=s}^{r_1}\mathrm{Re}f_j(x,y)$ and $\mathrm{Im}\,f(x,y) = \sum_{j=t}^{r_2}\mathrm{Im}f_j(x,y)$ have the forms of $\mathrm{Re}f_s(x,y) = (x^2 + y^2)^{p_1}$ and $\mathrm{Im}f_t(x,y) = (x^2+y^2)^{p_2}$ with $s = 2p_1 \geqslant 0$ and $t = 2p_2 \geqslant 0$, respectively.

Proof Since the conjugate $\bar{f}(x,y)$ is also an invariant algebraic curve with cofactor $\overline{L_f}(x,y)$, then $D((\mathrm{Re}f)^2 + (\mathrm{Im}f)^2) = 2((\mathrm{Re}f)^2 + (\mathrm{Im}f)^2)\mathrm{Re}L_f$. So $(\mathrm{Re}f)^2 + (\mathrm{Im}f)^2$ is an real invariant algebraic curve of system (2). From Theorem 2.1, $\mathrm{Re}L_f(0,0) = 0$. From equation (3), we have

$$\begin{cases} D(\mathrm{Re}f) = \mathrm{Re}f\mathrm{Re}L_f - \mathrm{Im}f\mathrm{Im}L_f \\ D(\mathrm{Im}f) = \mathrm{Re}f\mathrm{Im}L_f + \mathrm{Im}f\mathrm{Re}L_f \end{cases}. \tag{6}$$

If $L_f(0,0) = 0$, then $y\frac{\partial \mathrm{Re}\,f_s}{\partial x} - x\frac{\partial \mathrm{Re}\,f_s}{\partial y} = 0$ or $y\frac{\partial \mathrm{Im}\,f_t}{\partial x} - x\frac{\partial \mathrm{Im}\,f_t}{\partial y} = 0$. Hence this result is proved. This proof is analogous to the proof of Theorem 2.1.

Remark 2.1 Let $\mathrm{i} = \sqrt{-1}$. If $f(x,y) \in \mathbb{C}[x,y]$ is an invariant algebraic curve of system (2) with cofactor $L_f(x,y)$, then $\mathrm{i}f(x,y)$ is also an invariant curve with cofactor $L_f(x,y)$ and the real part is interchanged with the imaginary part.

Garcia and Grau[11] stated that, if an invariant algebraic curve $f(x,y)$, whose imaginary part is not null, appears in the expression of a first integral or integrating factors with exponent λ, then the conjugate $\bar{f}(x,y)$ appears in the same expression with exponent $\bar{\lambda}$. Then,

$$f^{\lambda}\bar{f}^{\bar{\lambda}} = ((\mathrm{Re}f)^2 + (\mathrm{Im}f)^2)^{\mathrm{Re}\lambda}\exp\left\{-2\mathrm{Im}\lambda\arctan\left(\frac{\mathrm{Im}f}{\mathrm{Re}f}\right)\right\}. \tag{7}$$

If cofactor $L_f(0,0) = 0$ and the lowest order terms $\mathrm{Re}f_s(x,y) \neq (x^2+y^2)^{p_1}$, based on Theorem 2.2 and remark 2.1 the lowest order terms of $\mathrm{Im}f$ has the form of $(x^2+y^2)^{p_2}$, that is to say, $\mathrm{Re}f$ or $\mathrm{Im}f$ are not equal to zero in some deleted neighborhood $U^\circ(O)$ of the origin. Since $f^{\lambda}\bar{f}^{\bar{\lambda}} = \exp(\pi\mathrm{Im}\lambda)(\mathrm{i}f)^{\lambda}\overline{(\mathrm{i}f)}^{\bar{\lambda}}$, the real function $f^{\lambda}\bar{f}^{\bar{\lambda}}$ is not identically zero and differentiable in some deleted neighborhood $U^\circ(O)$.

3 Exponential factor

Given two coprime polynomials $f, g \in k[x,y]$, the function $e = \exp(g/f)$ is called an exponential factor of the vector field $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ with the degree $m = \max(\deg(P),\deg(Q))$, if $X(e) = eL_e$, where L_e is a polynomial of degree at most $(m-1)$. The polynomial L_e is called the cofactor of the exponential factor. Proposition 3.1 on exponential factor is given by Pereira[9].

Proposition 3.1 If $e = \exp(g/f)$ is an exponential factor with cofactor L_e for the vector field D, then $f = 0$ is an invariant algebraic curve and g satisfies the equation

$$D(g) = gL_f + fL_e, \tag{8}$$

where L_f is the cofactor of f.

Remark 3.1 Let $f(x,y)$, $g(x,y) \in \mathbb{C}[x,y]$ and $\exp(g/f)$ be an exponential factor of the real system (2) with cofactor L_e. Since $D(\exp(g/f)) = D\left(\exp\left\{\frac{\operatorname{Re} g \operatorname{Re} f + \operatorname{Im} g \operatorname{Im} f}{(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2} + \mathrm{i}\frac{\operatorname{Im} g \operatorname{Re} f - \operatorname{Re} g \operatorname{Im} f}{(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2}\right\}\right) = \exp(g/f)(\operatorname{Re} L_e + \mathrm{i}\operatorname{Im} L_e)$, then $\exp\left\{\frac{\operatorname{Re} g \operatorname{Re} f + \operatorname{Im} g \operatorname{Im} f}{(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2}\right\}$ and $\exp\left\{\frac{\operatorname{Im} g \operatorname{Re} f - \operatorname{Re} g \operatorname{Im} f}{(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2}\right\}$ are also exponential factors of the real system (2) with cofactors $\operatorname{Re} L_e$ and $\operatorname{Im} L_e$.

Example 3.1 (see Ref. [7]) Consider the system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y + ax^3 + bx^2y + cy^3, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x - cx^3 + ax^2y + (b-2c)xy^2.\end{aligned} \tag{9}$$

System (9) has an invariant line given by $f(x,y) = x + \mathrm{i}y = 0$ with cofactors $L_f(x,y) = -\mathrm{i} + ax^2 + bxy - c(\mathrm{i}x^2 + xy + \mathrm{i}y^2)$. Since $D(1 + (b-c)x^2 - axy) = 2(1 + (b-c)x^2 - axy)(ax^2 + bxy - cxy) + (x^2 + y^2)(-a + acx^2 + 2c(b-c)xy - acy^2)$, the function $e = \exp\left\{\frac{1 + (b-c)x^2 - axy}{x^2 + y^2}\right\}$ is an exponential factor with cofactor $L_e = -a + acx^2 + 2c(b-c)xy - acy^2$.

If we have

$$\lambda_1 \operatorname{Re}(L_f) + \lambda_2 \operatorname{Im}(L_f) + \rho L_e = 0,$$

that is to say,

$$\begin{cases} a\lambda_1 + 2c\lambda_2 = 0 \\ (b-c)\lambda_1 + 2c(b-c)\rho = 0 \\ \lambda_2 = a\rho. \end{cases}$$

Then the function $F(x,y) = (f^{\lambda}\bar{f}^{\bar{\lambda}})^{\frac{1}{2}} e^{\rho} = \mathbf{C}$, where $\lambda = \lambda_1 + \mathrm{i}\lambda_2$, is the first integral of system (8).

4 Determining the types of equilibrium points

In order to construct the explicit first integral or integrating factors which can be used to distinguish the types of singularitie, we need to obtain more invariant curves. Christopher[8] used invariant curves, which did not vanish at the origin, in seeking local integrals or integrating factors. In this work, more invariant algebraic curves which may vanish at equilibrium point are considered. Sometime, the constructed function may be undefined at the singular point. Considering system (2), we know that the invariant algebraic curves of system (2) have the proposition in Theorems 2.1 and 2.2. We improve the methods of distinguishing between a center and a focus by using more invariant algebraic curves. Hence, we have the theorem given below.

Theorem 4.1 Suppose system (2) admits p distinct invariant algebraic curves $f_i = 0$ with cofactor L_{f_i} which vanishes at equilibrium point for each $i = 1, \cdots, p$, and q independent exponential factors e_j for $j = 1, \cdots, q$. If there exist $\lambda_i, \rho_j \in k$ not all zero such that $\sum_{i=1}^{p} \lambda_i L_{f_i} + \sum_{j=1}^{q} \rho_j L_{e_j} + \operatorname{div}(P,Q) = 0$, then the equilibrium point $O(0,0)$ is a center if and only if first form curve integrals $\oint_{x^2+y^2=r^2} (B(x,y)\overline{B(x,y)})^{\frac{1}{2}}(xP + yQ)\,\mathrm{d}s = 0$, where $r > 0$ is sufficiently small and $B(x,y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} e_1^{\rho_1} \cdots e_q^{\rho_q}$, is defined in a deleted neighborhood of the origin.

Proof Since $\sum_{i=1}^{p} \lambda_i L_{f_i} + \sum_{j=1}^{q} \rho_j L_{e_j} + \operatorname{div}(P,Q) = 0$, then $\sum_{i=1}^{p} \bar{\lambda}_i \bar{L}_{f_i} + \sum_{j=1}^{q} \bar{\rho}_j \bar{L}_{e_j} + \operatorname{div}(P,Q) = 0$, that is to say, $\overline{B(x,y)}$ is also an integrating factor of the real system (2). The complex invariant algebraic curves appear in the expression of a first integral or integrating factors with the conjugate curves appearing in the expression. The real function $f_i^{\lambda_i} \bar{f}_i^{\bar{\lambda}_i}$ is not identically zero and differentiable in some neighborhood $U^{\circ}(O)$. Based on Remark 3.1, exponential factor $e_j^{\lambda_j} \bar{e}_j^{\bar{\lambda}_j}$ has constant sign and is not identically zero in some deleted neighborhood $U^{\circ}(O)$ of the origin. So the real function $(B(x,y)\overline{B(x,y)})^{\frac{1}{2}}$ is not identically zero and differentiable in $U^{\circ}(O)$. Suppose that the origin is a focus. Take $x_1 > 0$ sufficiently small such that the arc of the positive semi-orbit from $(x_1, 0)$ to its next crossing of the positive x-axis $(x_2, 0)$ remains in $U^{\circ}(O)$. Applying Green's Theorem to the region bounded by this arc and the line segment from $(x_1, 0)$ to $(x_2, 0)$, the curve $C_1: x^2 + y^2 = r^2$, and taking $r > 0$ sufficiently small immediately leads to a contradiction.

Remark 4.1　Since

$$\left| \oint_{x^2+y^2=r^2} (B(x,y)\overline{B(x,y)})^{\frac{1}{2}} \left(P\frac{x}{r} + Q\frac{y}{r}\right) \mathrm{d}s \right| \leqslant \max_{x^2+y^2=r^2} \{2\pi (B(x,y)\overline{B(x,y)})^{\frac{1}{2}} |(xP + yQ)|\},$$

then the origin is a center if $(B(x,y)\overline{B(x,y)})^{\frac{1}{2}}(xP + yQ) = o(1)$ with $r = \sqrt{x^2 + y^2}$ sufficiently small. It is known that the degree of the lowest order terms in $xP + yQ$ is at least 3. So the result can be extended to the case that the function $(B(x,y)\overline{B(x,y)})^{\frac{1}{2}}$ has the proposition $\lim_{(x,y)\to(0,0)} (x^2 + y^2)^{\gamma} (B(x,y)\overline{B(x,y)})^{\frac{1}{2}} = 0$ where $\gamma < \frac{3}{2}$.

If $\sum_{i=1}^{p} \lambda_i L_{f_i} + \sum_{j=1}^{q} \rho_j L_{e_j} + \mathrm{div}(P,Q)$ has constant sign and is not identically zero on any open subset of a neighborhood of the origin, the next theorem is obtained by playing the Bendixson-Dulac criterion[12].

Theorem 4.2　Suppose system (2) admits p distinct invariant algebraic curves $f_i = 0$ with cofactor L_{f_i} which vanishes at equilibrium point for each $i = 1, \cdots, p$, and q independent exponential factor e_j with cofactor L_{e_j} for $j = 1, \cdots, q$. If there exist $\lambda_i, \rho_j \in k$ not all zero such that $\sum_{i=1}^{p} \lambda_i L_{f_i} + \sum_{j=1}^{q} \rho_j L_{e_j} + \mathrm{div}(P, Q)$ has constant sign and is not identically zero on any open subset of a neighborhood of the origin, then the equilibrium point is a focus.

Example 4.1　Consider the systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + axy, \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \frac{a-3}{2}x^2 + \frac{3a}{2}y^2. \quad (10)$$

The invariant algebraic curve $x^2 + y^2 + x^3$ vanishes at equilibrium point and the function $B(x, y) = (x^2 + y^2 + x^3)^{-\frac{4}{3}}$ is undefined at the origin O and an integrating factor of systems (10) in the deleted neighborhood $U^{\circ}(O)$. Based on Ramark 4.1, we have that the origin is a center. Then,

$$\oint_{x^2+y^2=r^2} (B(x,y)\overline{B(x,y)})^{\frac{1}{2}}(xP + yQ)\mathrm{d}s = \oint_{x^2+y^2=r^2} \frac{3}{2} \frac{(a-1)x^2y + ay^3}{(x^2 + y^2 + x^3)^{\frac{4}{3}}} \mathrm{d}s = 0.$$

Example 4.2　Consider the systems

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y + x^2y - xy^2 + y^3, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x - x^3 - xy^2 - y^3. \end{aligned} \quad (11)$$

Since $X(F) = -(x^2 + y^2)^2 + o(r^4)$ where $r = \sqrt{x^2 + y^2}$, then $F(x,y) = x^2 + y^2 + x^3y + xy^3$ is a Liapunov function of the system. So the origin is a focus. Based on Theorem 4.1, we can reach the same conclusion. Because the function $B(x,y) = (x^2 + y^2)^{-2}$ is the integrating factor of systems in the deleted neighborhood $U^{\circ}(O)$, $\oint_{x^2+y^2=r^2} (B(x,y)\overline{B(x,y)})^{\frac{1}{2}}(xP + yQ)\mathrm{d}s = \oint_{x^2+y^2=r^2} \frac{-y^2}{x^2+y^2}\mathrm{d}s \neq 0$.

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