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On the exponent of $N^m K_2(\mathbb{F}[C_{p^n}])$ *

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Abstract Let C_{p^n} be the cyclic p-group of order p^n and $\mathbb{F}$ a finite field of characteristic p. For any integer $1 \leqslant l \leqslant n$, we obtain infinitely many non-trivial elements of order p^l in $N^m K_2(\mathbb{F}[C_{p^n}])$. In fact, these elements form a generating set of $N^m K_2(\mathbb{F}[C_{p^n}])$ and the exponent of $N^m K_2(\mathbb{F}[C_{p^n}])$ is p^n.

Keywords K-theory; Bass Nil groups; truncated polynomial

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交换群 $N^m K_2(\mathbb{F}[C_{p^n}])$ 的指数

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摘 要 令 C_{p^n} 是阶为 p^n 的循环 p 群, $\mathbb{F}$ 是特征为 p 的有限域。对于任何整数 $1 \leqslant l \leqslant n$, 得到 $N^m K_2(\mathbb{F}[C_{p^n}])$ 中无限多个非平凡的 p^l 阶元素。事实上, 这些元素组成 $N^m K_2(\mathbb{F}[C_{p^n}])$ 的一个生成元集, 并且 $N^m K_2(\mathbb{F}[C_{p^n}])$ 的指数为 p^n。

关键词 K 理论;Bass Nil 群;截断多项式

Let R be a ring with unit. The Bass Nil groups $N^m K_i(R)$ are introduced by Bass[1] in order to investigate the relation between $K_i(R[x_1, \cdots, x_m])$ and $K_i(R)$. For any $i \in \mathbb{Z}$, $NK_i(R)$ is defined to be the kernel of surjective map $K_i(R[x_1]) \to K_i(R)$ induced by $x_1 \mapsto 0$. And $N^m K_i(R)$ is defined by iteration, i. e., the kernel of the surjection $N^{m-1}K_i(R[x_m]) \to N^{m-1}K_i(R)$ induced by $x_m \mapsto 0$. When $i = 0, 1, 2$, $K_i(R)$ are the classical algebraic K-groups defined by Grothendieck[2], Bass[1] and Milnor[3], respectively. When $i < 0$, the negative K-theory is defined by Bass[1]. When $i > 2$, $K_i(R) = \pi_i(K(R))$ is defined to be the i-th homotopy group of the K-theory space $K(R)$ which was first invented by Quillen[4] via plus-construction or Q-construction. As for the Bass Nil groups, the most known property is the following phenomenon.

Theorem A (See Refs. [5-9]). Let R be a ring. For any $i, m \in \mathbb{Z}$ and $m \geqslant 1$, if $N^m K_i(R) \neq 0$, then it is not finitely generated as an abelian group.

Let p be a prime number. In some cases, $NK_i(R)$ are abelian p-groups[8]. However, the

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exponents of these abelian p-groups are not completely determined. For example, the exponents of $NK_0(\mathbb{Z}[C_4])$ and $NK_1(\mathbb{Z}[C_4])$ are both 2, but the exponent of $NK_2(\mathbb{Z}[C_4])$ is still unknown[10].

Let $\mathbb{F}$ be a finite field of characteristic p and C_{p^n} the cyclic p-group of order p^n. Since $K_2(\mathbb{F}[C_{p^n}]) = K_2(\mathbb{F}) = 0$ (see Ref. [11]), we have

$$K_2(\mathbb{F}[C_{p^n}][x_1,\cdots,x_m]) \cong (1+N)^m K_2(\mathbb{F}[C_{p^n}]) = \bigoplus_{i=1}^{m}\binom{m}{i} N^i K_2(\mathbb{F}[C_{p^n}]).$$

In Ref. [12], Juan-Pineda showed the non-finiteness of $NK_2(\mathbb{F}_p[C_{p^n}])$ by giving one non-trivial element of order p and concluded that $NK_2(\mathbb{Z}[C]) \neq 0$ for any non-trivial cyclic group C. In this paper, we could give infinitely many non-trivial elements of order p^l for any $1 \leqslant l \leqslant n$ in $N^m K_2(\mathbb{F}[C_{p^n}])$. In fact, we give a presentation of $N^m K_2(\mathbb{F}[C_{p^n}])$ in terms of Dennis-Stein symbols and show that the exponent of $N^m K_2(\mathbb{F}[C_{p^n}])$ is p^n.

1 Main result

Let $\mathbb{F}$ be a finite field with p^f elements and $B = \{1, b, b^2, \cdots, b^{f-1}\}$ a basis of $\mathbb{F}$ as a vector space over the finite field $\mathbb{F}_p$ of p elements. Let C_{p^n} be the cyclic group of order $p^n (n \geqslant 1)$ generated by σ. Let $J = J(\mathbb{F}[C_{p^n}])$ be the Jacobson radical of the group algebra $\mathbb{F}[C_{p^n}]$. The notation $\oplus_\infty$ denotes a countably infinite direct sum, i.e., $\oplus_\infty = \oplus_{s \in S}$ for some countably infinite set S.

Lemma 1.1 $NK_2(\mathbb{F}[C_{p^n}]) \cong K_2(\mathbb{F}[C_{p^n}][x], J[x])$.

Proof Observe that $\mathbb{F}[C_{p^n}][x]/J[x] \cong \mathbb{F}[x]$ and $\mathbb{F}[x] \to \mathbb{F}[C_{p^n}][x]$ is a split inclusion. Since $\mathbb{F}$ is a finite field, $K_2(\mathbb{F}[C_{p^n}]) = K_2(\mathbb{F}) = 0$. And $K_2(\mathbb{F}[x]) = K_2(\mathbb{F}) \oplus NK_2(\mathbb{F}) = 0$ because $\mathbb{F}$ is regular. Hence the result follows from the two exact sequences of K-groups:

$$0 \to K_2(\mathbb{F}[C_{p^n}][x], J[x]) \to K_2(\mathbb{F}[C_{p^n}][x]) \to K_2(\mathbb{F}[C_{p^n}][x]/J[x]) = 0 \to 0,$$

$$0 \to NK_2(\mathbb{F}[C_{p^n}]) \to K_2(\mathbb{F}[C_{p^n}][x]) \to K_2(\mathbb{F}[C_{p^n}]) = 0 \to 0. \qquad \square$$

Let $I = (t_1^{p^n})$ be a proper ideal in the polynomial ring $\mathbb{F}[t_1, t_2, \cdots, t_{m+1}]$. Then

$$\mathbb{F}[C_{p^n}] \cong \mathbb{F}[t_1]/(t_1^{p^n}),$$

$$\mathbb{F}[C_{p^n}][x_1, \cdots, x_m] \cong \mathbb{F}[t_1, \cdots, t_{m+1}]/I,$$

via $\sigma - 1 \mapsto t_1$ and $x_i \mapsto t_{i+1}$. Let $A = \mathbb{F}[t_1, \cdots, t_{m+1}]/I$ and $M = (\overline{t_1})$ be its nilradical where $\overline{t_1} = t_1 + I$. Then $\mathbb{F}[x_1, \cdots, x_m] \cong A/M$ and $K_2(A) \cong K_2(A, M)$. So the above lemma becomes $NK_2(\mathbb{F}[C_{p^n}]) \cong K_2(\mathbb{F}[t_1, t_2]/I, M)$ $(m = 1)$.

Lemma 1.2 Let $1 \leqslant t \leqslant n-1$ and $1 \leqslant h < p$ be integers. If $p^{n-t-1} < k \leqslant p^{n-t}$, $\left\lceil \log_p \frac{p^n+1}{pk} \right\rceil = \left\lceil \log_p \frac{p^n}{pk-h} \right\rceil = t$, and if $k = 1$, $\left\lceil \log_p \frac{p^n+1}{pk} \right\rceil = \left\lceil \log_p \frac{p^n}{pk-h} \right\rceil = n$, where $\lceil a \rceil = \min\{s \in \mathbb{Z} \mid s \geqslant a\}$ denotes the smallest integer no less than a.

Proof If $k = 1$, the computation is easy. Suppose $p^{n-t-1} + 1 \leqslant k \leqslant p^{n-t}$, the result follows from the inequalities

$$p^{t-1} < \frac{p^n}{p^{n-t+1} - h} \leqslant \frac{p^n}{pk - h} \leqslant \frac{p^n}{p^{n-t} + p - h} < p^t,$$

$$p^{t-1} = \frac{p^n}{p^{n-t+1}} < \frac{p^n+1}{pk} \leqslant \frac{p^n+1}{p^{n-t}+p} < p^t. \qquad \square$$

Theorem 1.1 Let C_{p^n} be the cyclic group of order p^n $(n \geqslant 1)$ generated by σ. For any integer $m \geqslant 1$, $N^m K_2(\mathbb{F}[C_{p^n}]) \cong \oplus_\infty (\bigoplus_{i=1}^{n} \mathbb{Z}/p^i\mathbb{Z})$ can be generated by these elements:

the generators of order $p^l (1 \leqslant l \leqslant n-1)$ are $\langle b(\sigma - 1)^{pk-1} \prod_{j=1}^{m} x_j^{\beta_j}, (\sigma - 1) \rangle$, where $p^{n-l-1} < k \leqslant p^{n-l}$ and $\gcd(p, \beta_1, \cdots, \beta_m) = 1$, $\langle b(\sigma - 1)^{pk} (\prod_{j=1}^{m} x_j^{\beta_j})/x_i, x_i \rangle$, where $1 \leqslant i \leqslant m$, $p^{n-l-1} \leqslant k < p^{n-l}$, $\gcd(p, \beta_1, \cdots, \beta_m) = 1$ and $i \neq \min\{j \mid \beta_j \not\equiv 0 \bmod p\}$, $\langle b(\sigma - 1)^{pk-h} (\prod_{j=1}^{m} x_j^{\beta_j})/x_i, x_i \rangle$, where $1 \leqslant i \leqslant m$, $p^{n-l-1} < k \leqslant p^{n-l}$, $1 \leqslant h < p$; and generators of order p^n are $\langle b(\sigma - 1)^{p-1} \prod_{j=1}^{m} x_j^{\beta_j}, (\sigma - 1) \rangle$, where $\gcd(p, \beta_1, \cdots, \beta_m) = 1$, $\langle b(\sigma - 1)^{h} (\prod_{j=1}^{m} x_j^{\beta_j})/x_i, x_i \rangle$, where $1 \leqslant i \leqslant m$ and $1 \leqslant h < p$.

For all the above symbols, $b \in B$ and $\beta = (\beta_1, \cdots, \beta_m) \in \mathbb{N}_+^m$, where $\mathbb{N}_+$ is the set of positive integers.

Proof Suppose we get a generating set of $K_2(\mathbb{F}[C_{p^n}][x_1,\cdots,x_m]) \cong K_2(A,M)$ in terms of Dennis-Stein symbols. Fix j different indeterminates in $\{x_1,\cdots,x_m\}$, say $\{x_{i_1},\cdots,x_{i_j}\}$. The elements of the direct summand $N^j K_2(\mathbb{F}[C_{p^n}]) \subset K_2(\mathbb{F}[C_{p^n}][x_{i_1},\cdots,x_{i_j}]) \subset K_2(\mathbb{F}[C_{p^n}][x_1,\cdots,x_m])$ can be represented by using those Dennis-Stein symbols containing these j different indeterminates. Hence $K_2(\mathbb{F}[C_{p^n}][x_1,\cdots,x_m])$ contains $\binom{m}{j}$ pieces of $N^j K_2(\mathbb{F}[C_{p^n}])$. So the elements of $N^m K_2(\mathbb{F}[C_{p^n}])$ can be represented by using those Dennis-Stein symbols in $K_2(\mathbb{F}[C_{p^n}][x_1,\cdots,x_m])$ containing all the m indeterminates.

We follow the notations in Ref. [13]. Let $\mathbb{N} = \{0,1,2\cdots\}$ be the set of non-negative integers and $\mathbb{N}_+ = \{1,2,3,\cdots\}$ the set of positive integers. Let $\varepsilon^i = (0,\cdots,0,1,0,\cdots,0) \in \mathbb{N}^{m+1}$ be the i-th basis vector. For $\alpha \in \mathbb{N}^{m+1}$, write $t^\alpha = t_1^{\alpha_1}\cdots t_{m+1}^{\alpha_{m+1}}$ where $t_1 = \sigma - 1$ and $t_{i+1} = x_i$ for $1 \leqslant i \leqslant m$. Define

$$\Delta' = \{\alpha = (\alpha_1,\cdots,\alpha_{m+1}) \in \mathbb{N}_+^{m+1} \mid \alpha_1 \geqslant p^n\},$$

$$\Lambda' = \{(\alpha,i) \in \mathbb{N}_+^{m+1} \times \{1,2,\cdots,m+1\}\}.$$

For $(\alpha,i) \in \Lambda'$, let $[\alpha,i] = \min\{k \in \mathbb{Z} \mid k\alpha - \varepsilon^i \in \Delta'\}$ and $w(\alpha,i) = \min\{w \in \mathbb{N} \mid p^w \geqslant [\alpha,i]\}$. Then $[\alpha,1] = \left\lceil \frac{p^n+1}{\alpha_1} \right\rceil$, $[\alpha,j] = \left\lceil \frac{p^n}{\alpha_1} \right\rceil$ for any $j \neq 1$. If $\gcd(p,\alpha_1,\cdots,\alpha_{m+1}) = 1$, put $[\alpha] = \max\{[\alpha,i] \mid \alpha_i \not\equiv 0 \bmod p\}$. Set $\Lambda'^0 = \{(\alpha,i) \in \Lambda' \mid \gcd(p,\alpha_1,\cdots,\alpha_{m+1}) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha,j] = [\alpha]\}\}$, and let $\Lambda'^0_{>1} = \{(\alpha,i) \in \Lambda'^0 \mid [\alpha,i] > 1\}$.

Then by Corollary 2.7 in Ref. [13] and the above discussion, $N^m K_2(\mathbb{F}[C_{p^n}])$ has a presentation with

generators: $\langle bt^{\alpha-\varepsilon^i}, t_i\rangle$, where $b \in B$, $(\alpha,i) \in \Lambda'^0_{>1}$;

relations: $p^{w(\alpha,i)}\langle bt^{\alpha-\varepsilon^i}, t_i\rangle = 0$, where $w(\alpha,i) = \lceil \log_p[\alpha,i] \rceil$.

It is sufficient to determine the set $\Lambda'^0_{>1}$.

If $\alpha_1 = p^n$ and at least one of α_j with $p \nmid \alpha_j$, then only $(\alpha,1) \in \Lambda'^0_{>1}$ and $[\alpha,1] = 2$.

If $\alpha_1 = pk$ for some $1 \leqslant k < p^{n-1}$ and j is the smallest number such that $p \nmid \alpha_j$, i.e., $p \mid \alpha_1,\cdots,p \mid \alpha_{j-1}$ and $p \nmid \alpha_j$, then all (α,i) except (α,j) are in $\Lambda'^0_{>1}$.

If $p \nmid \alpha_1$, i.e., $\alpha_1 = pk - h$ for some $1 \leqslant k \leqslant p^{n-1}$ and $1 \leqslant h < p$, then for each $j \geqslant 2$, $(\alpha,j) \in \Lambda'^0_{>1}$.

So one gets

$$\Lambda'^0_{>1} = \{(\alpha,1) \mid \alpha_1 = p^n, \gcd(p,\alpha_2,\cdots,\alpha_{m+1}) = 1\} \cup \left\{(\alpha,i) \,\middle|\, \begin{array}{c} \alpha_1 = pk, 1 \leqslant k < p^{n-1}, \\ \gcd(p,\alpha_2,\cdots,\alpha_{m+1}) = 1, \\ i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p\} \end{array}\right\} \cup \left(\bigcup_{j=2}^{m+1}\left\{(\alpha,j) \,\middle|\, \begin{array}{c} \alpha_1 = pk - h, \\ 1 \leqslant k \leqslant p^{n-1}, \\ 1 \leqslant h < p \end{array}\right\}\right).$$

Let $b \in B$. For any $\beta \in \mathbb{N}_+^m$, write $x^\beta = x_1^{\beta_1}x_2^{\beta_2}\cdots x_m^{\beta_m}$. We can get a presentation of $N^m K_2(\mathbb{F}[C_{p^n}])$ in terms of the following Dennis-Stein symbols with generators

- $\langle b(\sigma-1)^{pk-1}x^\beta, (\sigma-1)\rangle$ where $1 \leqslant k \leqslant p^{n-1}$ and $\gcd(p,\beta_1,\cdots,\beta_m) = 1$;
- $\langle b(\sigma-1)^{pk}x^{\beta-\varepsilon^i}, x_i\rangle$ where $1 \leqslant k < p^{n-1}$, $\gcd(p,\beta_1,\cdots,\beta_m) = 1$ and $i \neq \min\{j \mid \beta_j \not\equiv 0 \bmod p\}$;
- $\langle b(\sigma-1)^{pk-h}x^{\beta-\varepsilon^i}, x_i\rangle$ where $1 \leqslant k \leqslant p^{n-1}$, $1 \leqslant h < p$, $1 \leqslant i \leqslant m$.

The relations are

- $p^{\lceil \log_p \frac{p^n+1}{pk} \rceil}\langle b(\sigma-1)^{pk-1}x^\beta, (\sigma-1)\rangle = 0$,
- $p^{\lceil \log_p \frac{p^n}{pk} \rceil}\langle b(\sigma-1)^{pk}x^{\beta-\varepsilon^i}, x_i\rangle = 0$,
- $p^{\lceil \log_p \frac{p^n}{pk-h} \rceil}\langle b(\sigma-1)^{pk-h}x^{\beta-\varepsilon^i}, x_i\rangle = 0$.

Then by Lemma 1.2, the result follows. □

2 Examples

Example 2.1 Let C_4 be the cyclic group of order 4 generated by σ. Then $NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})$ can be generated by these elements: the generators of order 4 are

$$\{\langle(\sigma-1)x^{i-1}, x\rangle \mid i \geqslant 1\},$$
$$\{\langle(\sigma-1)x^{2i-1}, (\sigma-1)\rangle \mid i \geqslant 1\},$$

and the generators of order 2 are

$$\{\langle(\sigma-1)^3x^{i-1}, x\rangle \mid i \geqslant 1\},$$
$$\{\langle(\sigma-1)^3x^{2i-1}, (\sigma-1)\rangle \mid i \geqslant 1\}.$$

$N^2K_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})$ can be generated by these elements: the generators of order 4 are

$$\{\langle(\sigma-1)x^{i-1}y^{j},x\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)x^{i}y^{j-1},y\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)x^{2i-1}y^{j},(\sigma-1)\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)x^{2i}y^{2j-1},(\sigma-1)\rangle \mid i\geqslant 1,j\geqslant 1\},$$

and the generators of order 2 are

$$\{\langle(\sigma-1)^{3}x^{i-1}y^{j},x\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)^{3}x^{i}y^{j-1},y\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)^{2}x^{2i-1}y^{2j-1},x\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)^{2}x^{2i-1}y^{j-1},y\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)^{3}x^{2i-1}y^{j},(\sigma-1)\rangle \mid i\geqslant 1,j\geqslant 1\},$$
$$\{\langle(\sigma-1)^{3}x^{2i}y^{2j-1},(\sigma-1)\rangle \mid i\geqslant 1,j\geqslant 1\},$$

where x,y are indeterminates.

Corollary 2.1 $NK_1(\mathbb{Z}[C_{p^2}])\cong\bigoplus_{\infty}\mathbb{Z}/p\mathbb{Z}$.

Proof Assume C_{p^2} is generated by σ. There is a Milnor square,

$$\begin{array}{ccc}\mathbb{Z}[C_{p^2}] & \xrightarrow{\sigma\mapsto\zeta_{p^2}} & \mathbb{Z}[\zeta_{p^2}] \\ {\scriptstyle \sigma\mapsto\sigma^p}\downarrow & & \downarrow \\ \mathbb{Z}[C_p] & \longrightarrow & \mathbb{F}_p[C_p]\end{array}$$

where ζ_{p^2} is a primitive p^2-th root of unity and $\mathbb{Z}[\zeta_{p^2}]$ is the ring of integers in $\mathbb{Q}(\zeta_{p^2})$. By the Mayer-Vietoris sequence for NK-functors, we get an exact sequence

$$NK_2(\mathbb{Z}[C_{p^2}])\to NK_2(\mathbb{Z}[\zeta_{p^2}])\oplus NK_2(\mathbb{Z}[C_p])\to NK_2(\mathbb{F}_p[C_p])\to NK_1(\mathbb{Z}[C_{p^2}])\to NK_1(\mathbb{Z}[\zeta_{p^2}])\oplus NK_1(\mathbb{Z}[C_p]).$$

Since $\mathbb{Z}[\zeta_{p^2}]$ is regular, $NK_n(\mathbb{Z}[\zeta_{p^2}])=0$ for all n. The order of C_p is square-free. Hence $NK_1(\mathbb{Z}[C_p])=0$ (see Ref. [14]). And $NK_2(\mathbb{Z}[C_p])\cong\bigoplus_{\infty}\mathbb{Z}/p\mathbb{Z}$ (see Ref. [15]). So the above exact sequence becomes

$$NK_2(\mathbb{Z}[C_p])\to NK_2(\mathbb{F}_p[C_p])\to NK_1(\mathbb{Z}[C_{p^2}])\to 0.$$

Moreover, we have $NK_2(\mathbb{F}_p[C_p])\cong\bigoplus_{\infty}\mathbb{Z}/p\mathbb{Z}$ and $NK_1(\mathbb{Z}[C_{p^2}])\neq 0$ (see Ref. [14]). Hence $NK_1(\mathbb{Z}[C_{p^2}])$ is not finitely generated, therefore $NK_1(\mathbb{Z}[C_{p^2}])\cong\bigoplus_{\infty}\mathbb{Z}/p\mathbb{Z}$. □

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