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因子 von Neumann 代数上的非线性混合 Lie 三重可导映射

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摘 要 本文通过经典的可导映射, 运用矩阵分块的方法, 证明了因子 von Neumann 代数 $\mathscr{A}$ 上的每一个非线性混合 Lie 三重可导映射都是可加的 $*$- 导子.

关键词 混合 Lie 三重可导映射; von Neumann 代数; $*$- 导子
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Nonlinear Mixed Lie Triple Derivable Mappings on Factor von Neumann Algebras

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Abstract We prove that every nonlinear mixed Lie triple derivable mapping from any factor von Neumann algebra $\mathscr{A}$ into itself is an additive $*$-derivation, with the classical derivable mapping and matrix block.

Keywords mixed Lie triple derivable mapping; von Neumann algebra; $*$-derivation
MR(2010) Subject Classification 47B47, 46L10
Chinese Library Classification O177.1

1 引言

设 $\mathscr{A}$ 是一个 $*$- 代数, $A,B\in\mathscr{A}$. 称 $[A,B]=AB-BA$, $[A,B]_*=AB-BA^*$ 分别为 A 与 B 的 Lie 积和斜 Lie 积. 设 $\delta:\mathscr{A}\to\mathscr{A}$ 是一个映射 (不必可加或线性). 如果对任意的 $A,B\in\mathscr{A}$, 分别有 $\delta([A,B])=[\delta(A),B]+[A,\delta(B)]$ 和 $\delta([A,B]_*)=[\delta(A),B]_*+[A,\delta(B)]_*$, 则分别称 δ 是 $\mathscr{A}$ 上的非线性 Lie 可导映射和非线性斜 Lie 可导映射. 如果对任意的 $A,B,C\in$

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$\mathscr{A}$, 分别有 $\delta([[A,B],C]) = [[\delta(A),B],C] + [[A,\delta(B)],C] + [[A,B],\delta(C)]$ 和 $\delta([[A,B]_*,C]_*) = [[\delta(A),B]_*,C]_* + [[A,\delta(B)]_*,C]_* + [[A,B]_*,\delta(C)]_*$, 则分别称 δ 是 $\mathscr{A}$ 上的非线性 Lie 三重可导映射和非线性斜 Lie 三重可导映射. 显然, 非线性 Lie 可导映射和非线性斜 Lie 可导映射分别是非线性 Lie 三重可导映射和非线性斜 Lie 三重可导映射, 反之不成立. 如果对任意的 $A,B,C \in \mathscr{A}$, 有 $\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)]_*,C] + [[A,B]_*,\delta(C)]$, 则称 δ 是 $\mathscr{A}$ 上的非线性混合 Lie 三重可导映射. 一般地, 非线性混合 Lie 三重可导映射既不同于非线性 Lie 三重可导映射又不同于非线性斜 Lie 三重可导映射. 例如, 设 $\mathscr{A} = \mathbb{C} \oplus \mathbb{C}$, 映射 $\delta: \mathscr{A} \to \mathscr{A}$ 为

$$\delta(A) = \begin{cases} 0, & A = 0, \\ A + I, & A \neq 0. \end{cases}$$

则 δ 是 $\mathscr{A}$ 上的非线性混合 Lie 三重可导映射但不是非线性斜 Lie 三重可导映射. 设 $\mathscr{A} = M_2(\mathbb{C})$, 映射 $\delta: \mathscr{A} \to \mathscr{A}$ 为

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}(a_{11}+a_{22}) & 0 \\ 0 & a_{11}(a_{11}+a_{22}) \end{pmatrix},$$

则 δ 是 $\mathscr{A}$ 上的非线性 Lie 三重可导映射但不是非线性混合 Lie 三重可导映射.

环和代数上的非线性 Lie 可导映射和 Lie 三重可导映射的结构研究已获得了一系列结果 (见文 [1–4, 6, 9, 11, 13]). 近年来, 关于 $*$- 代数上的非线性斜 Lie 可导映射和非线性斜 Lie 三重可导映射的研究也引起了许多学者的关注, 并得到了一些有趣的结论 [5, 10, 12]. 如上所述, $*$- 代数上的非线性混合 Lie 三重可导映射既不同于非线性 Lie 三重可导映射又不同于非线性斜 Lie 三重可导映射, 本文将研究因子 von Neumann 代数上的非线性混合 Lie 三重可导映射.

设 $\mathscr{H}$ 是一个复 Hilbert 空间, $B(\mathscr{H})$ 是 $\mathscr{H}$ 上的所有有界线性算子全体, $\mathscr{A} \subseteq B(\mathscr{H})$ 是一个 von Neumann 代数. 称 $\mathscr{A}$ 为一个因子是指它的中心为 $\mathbb{C}I$. 我们知道因子 von Neumann 代数 $\mathscr{A}$ 一定是素代数. 即 $X, Y \in \mathscr{A}$ 且 $X\mathscr{A}Y = \{0\}$ 蕴含 $X = 0$ 或 $Y = 0$.

2 主要定理及其证明

本文主要得到下列结果:

定理 2.1 设 $\mathscr{A}$ 是复 Hilbert 空间上的因子 von Neumann 代数且 $\dim(\mathscr{A}) > 1$. 如果映射 $\delta: \mathscr{A} \to \mathscr{A}$ 满足对任意的 $A,B,C \in \mathscr{A}$, 有

$$\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)]_*,C] + [[A,B]_*,\delta(C)],$$

则 δ 是可加的 $*$- 导子.

为了证明定理 2.1, 需要以下引理.

引理 2.1[7] 设 $\mathscr{R}$ 为一个素环, $d: \mathscr{R} \to \mathscr{R}$ 是一个可加导子. 如果对任意的 $x \in \mathscr{R}$, 有 $xd(x) - d(x)x = 0$, 则 $\mathscr{R}$ 可交换或 $d = 0$.

引理 2.2 设 $\mathscr{A}$ 是因子 von Neumann 代数, $Y \in \mathscr{A}$. 如果对任意的 $B \in \mathscr{A}$, 有 $[Y,B]_* \in \mathbb{C}I$, 则 $Y \in \mathbb{C}I$.

证明 如果 $\dim(\mathscr{A}) = 1$, 则 $\mathscr{A} = \mathbb{C}I$. 从而 $Y \in \mathbb{C}I$. 如果 $\dim(\mathscr{A}) > 1$, 则 $\mathscr{A}$ 是非交换代数. 令 $h(B) = [Y,B]_*$, 则 $h(I) = Y - Y^* \in \mathbb{C}I$, 从而

$$h(B) = YB - B(Y - h(I)) = YB - BY + h(I)B.$$

定义映射 $g:\mathscr{A}\to\mathscr{A}$ 为 $g(B)=[Y,B]$, 则 g 是线性导子, 并且对任意的 $B\in\mathscr{A}$, 有 $Bg(B)-g(B)B=0$. 由引理 2.1 和 $\mathscr{A}$ 的非交换性, 从而对任意的 $B\in\mathscr{A}$, 有 $YB=BY$. 因此 $Y\in\mathbb{C}I$. 证毕.

以下设 $\mathscr{A}$ 是因子 von Neumann 代数且 $\dim(\mathscr{A})>1$, δ 为 $\mathscr{A}$ 上的非线性混合 Lie 三重可导映射, $P_1\in\mathscr{A}$ 为一个固定非平凡投影, $P_2=I-P_1$, $\mathscr{A}_{jk}=P_j\mathscr{A}P_k$, $j,k=1,2$.

引理 2.3[4] 设 $A_{jj}\in\mathscr{A}_{jj}$, $j=1,2$. 如果对任意的 $B_{12}\in\mathscr{A}_{12}$, 有 $A_{11}B_{12}=B_{12}A_{22}$, 则 $A_{11}+A_{22}\in\mathbb{C}I$.

引理 2.4 $P_1\delta(P_2)^*P_2=-P_1\delta(P_1)P_2$; $P_k\delta(P_k)P_j+P_k\delta(P_j)P_j=0$ $(1\leq j\neq k\leq 2)$.

证明 直接验证可得 $\delta(0)=0$. 从而对 $j,k\in\{1,2\}$ 且 $j\neq k$, 有

$$\begin{aligned}0=\delta([[P_k,P_j]_*,P_k])&=[[\delta(P_k),P_j]_*,P_k]+[[P_k,\delta(P_j)]_*,P_k]\\&=-P_j\delta(P_k)^*P_k-P_k\delta(P_k)P_j-P_j\delta(P_j)P_k-P_k\delta(P_j)P_j.\end{aligned}$$

对上式左乘 P_j 右乘 P_k 得 $P_1\delta(P_2)^*P_2=-P_1\delta(P_1)P_2$; 左乘 P_k 右乘 P_j 得 $P_k\delta(P_k)P_j+P_k\delta(P_j)P_j=0$. 证毕.

注 2.1 令 $T=P_1\delta(P_1)P_2+P_2\delta(P_2)P_1$, 则由引理 2.4, $T^*=-T$. 定义映射 $\Phi:\mathscr{A}\to\mathscr{A}$ 为 $\Phi(A)=\delta(A)-[A,T]$. 直接验证可知 $\Phi([[A,B]_*,C])=[[\Phi(A),B]_*,C]+[[A,\Phi(B)]_*,C]+[[A,B]_*,\Phi(C)]$ 对任意的 $A,B,C\in\mathscr{A}$ 成立. 再由引理 2.4, 当 $j,k\in\{1,2\}$ 且 $j\neq k$ 时, 有

$$\Phi(P_j)=\delta(P_j)-[P_j,T]=\delta(P_j)-P_j\delta(P_j)P_k+P_k\delta(P_k)P_j=P_j\delta(P_j)P_j+P_k\delta(P_j)P_k.\tag{2.1}$$

引理 2.5 Φ 是可加的.

证明 通过以下几个断言证明.

断言 1 对任意的 $A_{12}\in\mathscr{A}_{12},B_{21}\in\mathscr{A}_{21}$, 有 $\Phi(A_{12}+B_{21})=\Phi(A_{12})+\Phi(B_{21})$.

设 $T=\Phi(A_{12}+B_{21})-\Phi(A_{12})-\Phi(B_{21})$. 对任意的 $C_{21}\in\mathscr{A}_{21}$, 由 $[[P_2,C_{21}]_*,B_{21}]=0$ 和 $\Phi(0)=0$ 知

$$\begin{aligned}\Phi([[P_2,C_{21}]_*,A_{12}+B_{21}])&=\Phi([[P_2,C_{21}]_*,A_{12}])+\Phi([[P_2,C_{21}]_*,B_{21}])\\&=[[\Phi(P_2),C_{21}]_*,A_{12}+B_{21}]+[[P_2,\Phi(C_{21})]_*,A_{12}+B_{21}]\\&\quad+[[P_2,C_{21}]_*,\Phi(A_{12})+\Phi(B_{21})];\end{aligned}$$

另一方面, 有

$$\begin{aligned}\Phi([[P_2,C_{21}]_*,A_{12}+B_{21}])&=[[\Phi(P_2),C_{21}]_*,A_{12}+B_{21}]+[[P_2,\Phi(C_{21})]_*,A_{12}+B_{21}]\\&\quad+[[P_2,C_{21}]_*,\Phi(A_{12}+B_{21})].\end{aligned}$$

于是, 对任意的 $C_{21}\in\mathscr{A}_{21}$,

$$[[P_2,C_{21}]_*,T]=0.\tag{2.2}$$

对 (2.2) 式左乘 P_1 得 $P_1TC_{21}=0$. 从而由 $\mathscr{A}$ 的素性, 有 $T_{12}=0$. 类似可证 $T_{21}=0$.

$$\begin{aligned}\Phi(A_{12}+B_{21})&=\Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,A_{12}-B_{21}\right]\right)\\&=\left[\left[\Phi\left(\frac{\mathrm{i}}{2}P_1\right),\mathrm{i}(P_2-P_1)\right]_*,A_{12}-B_{21}\right]+\left[\left[\frac{\mathrm{i}}{2}P_1,\Phi(\mathrm{i}(P_2-P_1))\right]_*,A_{12}-B_{21}\right]\\&\quad+\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,\Phi(A_{12}-B_{21})\right]\end{aligned}$$

$$
\begin{aligned}
&= \Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2-P_1)\right]_*, A_{12}\right]\right) - \left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2-P_1)\right]_*, \Phi(A_{12})\right] \\
&\quad + \Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2-P_1)\right]_*, -B_{21}\right]\right) - \left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2-P_1)\right]_*, \Phi(-B_{21})\right] \\
&\quad + \left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2-P_1)\right]_*, \Phi(A_{12}-B_{21})\right] \\
&= \Phi(A_{12}) + \Phi(B_{21}) + [P_1, \Phi(A_{12}-B_{21}) - \Phi(A_{12}) - \Phi(-B_{21})].
\end{aligned}
$$

于是 $T = [P_1, \Phi(A_{12}-B_{21}) - \Phi(A_{12}) - \Phi(-B_{21})]$. 从而 $T_{11} = T_{22} = 0$.

断言 2 对任意的 $A_{jj} \in \mathscr{A}_{jj}$, $B_{jk} \in \mathscr{A}_{jk}$, $B_{kj} \in \mathscr{A}_{kj}$, $1 \le j \ne k \le 2$, 有

(a) $\Phi(A_{jj} + B_{jk}) = \Phi(A_{jj}) + \Phi(B_{jk})$;

(b) $\Phi(A_{jj} + B_{kj}) = \Phi(A_{jj}) + \Phi(B_{kj})$.

设 $T = \Phi(A_{jj} + B_{jk}) - \Phi(A_{jj}) - \Phi(B_{jk})$. 由 $[[P_j, A_{jj}]_*, P_j] = 0$ 和 $\Phi(0) = 0$ 知

$$
\begin{aligned}
\Phi([[P_j, A_{jj} + B_{jk}]_*, P_j]) &= \Phi([[P_j, A_{jj}]_*, P_j]) + \Phi([[P_j, B_{jk}]_*, P_j]) \\
&= [[\Phi(P_j), A_{jj} + B_{jk}]_*, P_j] + [[P_j, \Phi(A_{jj}) + \Phi(B_{jk})]_*, P_j] \\
&\quad + [[P_j, A_{jj} + B_{jk}]_*, \Phi(P_j)];
\end{aligned}
$$

另一方面

$$
\begin{aligned}
\Phi([[P_j, A_{jj} + B_{jk}]_*, P_j]) &= [[\Phi(P_j), A_{jj} + B_{jk}]_*, P_j] + [[P_j, \Phi(A_{jj} + B_{jk})]_*, P_j] \\
&\quad + [[P_j, A_{jj} + B_{jk}]_*, \Phi(P_j)].
\end{aligned}
$$

于是

$$
[[P_j, T]_*, P_j] = 0. \tag{2.3}
$$

对 (2.3) 式左乘 P_k 右乘 P_j, 左乘 P_j 右乘 P_k, 则 $T_{kj} = T_{jk} = 0$. 从而 $T = T_{jj} + T_{kk}$.

对任意的 $C_{jk} \in \mathscr{A}_{jk}$, 由 $[[C_{jk}, A_{jj}]_*, P_k] = 0$ 知

$$
\begin{aligned}
\Phi([[C_{jk}, A_{jj} + B_{jk}]_*, P_k]) &= \Phi([[C_{jk}, A_{jj}]_*, P_k]) + \Phi([[C_{jk}, B_{jk}]_*, P_k]) \\
&= [[\Phi(C_{jk}), A_{jj} + B_{jk}]_*, P_k] + [[C_{jk}, \Phi(A_{jj}) + \Phi(B_{jk})]_*, P_k] \\
&\quad + [[C_{jk}, A_{jj} + B_{jk}]_*, \Phi(P_k)];
\end{aligned}
$$

另一方面

$$
\begin{aligned}
\Phi([[C_{jk}, A_{jj} + B_{jk}]_*, P_k]) &= [[\Phi(C_{jk}), A_{jj} + B_{jk}]_*, P_k] + [[C_{jk}, \Phi(A_{jj} + B_{jk})]_*, P_k] \\
&\quad + [[C_{jk}, A_{jj} + B_{jk}]_*, \Phi(P_k)].
\end{aligned}
$$

于是 $[[C_{jk}, T]_*, P_k] = 0$. 对上式左乘 P_j 右乘 P_k, 则 $C_{jk} T P_k = 0$. 从而由 $\mathscr{A}$ 的素性, $T_{kk} = 0$. 由 $[[B_{jk}, C_{jk}]_*, P_k] = 0$ 易验证 $[[T, C_{jk}]_*, P_k] = 0$. 从而由 $\mathscr{A}$ 的素性, $T_{jj} = 0$. 所以 $\Phi(A_{jj} + B_{jk}) = \Phi(A_{jj}) + \Phi(B_{jk})$.

类似地, 可得 (b) 也成立.

断言 3 对任意的 $A_{jk}, B_{jk} \in \mathscr{A}_{jk}, 1 \le j \ne k \le 2$, 有 $\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk})$.

一方面, 由断言 1 得

$$
\Phi([[P_j + B_{jk}^*, P_j - A_{jk}]_*, P_j]) = \Phi(B_{jk}^* + A_{jk} + B_{jk}) = \Phi(B_{jk}^*) + \Phi(A_{jk} + B_{jk});
$$

另一方面, 由断言 1, 2 和 $\Phi(0)=0$, 得

$$\begin{aligned}
\Phi([[P_j+B_{jk}^*,P_j-A_{jk}]_*,P_j]) &= [[\Phi(P_j+B_{jk}^*),P_j-A_{jk}]_*,P_j] \\
&\quad + [[P_j+B_{jk}^*,\Phi(P_j-A_{jk})]_*,P_j] + [[P_j+B_{jk}^*,P_j-A_{jk}]_*,\Phi(P_j)] \\
&= [[\Phi(P_j)+\Phi(B_{jk}^*),P_j-A_{jk}]_*,P_j] \\
&\quad + [[P_j+B_{jk}^*,\Phi(P_j)+\Phi(-A_{jk})]_*,P_j]+[[P_j+B_{jk}^*,P_j-A_{jk}]_*,\Phi(P_j)] \\
&= \Phi([[P_j,P_j]_*,P_j]) + \Phi([[P_j,-A_{jk}]_*,P_j]) \\
&\quad + \Phi([[B_{jk}^*,P_j]_*,P_j]) + \Phi([[B_{jk}^*,-A_{jk}]_*,P_j]) \\
&= \Phi(A_{jk}) + \Phi(B_{jk}^*+B_{jk}) \\
&= \Phi(A_{jk}) + \Phi(B_{jk}^*) + \Phi(B_{jk}).
\end{aligned}$$

从而 $\Phi(A_{jk}+B_{jk})=\Phi(A_{jk})+\Phi(B_{jk})$.

断言 4　对任意的 $A_{jj},B_{jj}\in\mathscr{A}_{jj}$ $(j=1,2)$, 有 $\Phi(A_{jj}+B_{jj})=\Phi(A_{jj})+\Phi(B_{jj})$.

设 $T=\Phi(A_{jj}+B_{jj})-\Phi(A_{jj})-\Phi(B_{jj})$. 由 $\Phi(0)=0$ 及 $[[\mathrm{i}P_k,I]_*,A_{jj}+B_{jj}]=[[\mathrm{i}P_k,I]_*,A_{jj}]=[[\mathrm{i}P_k,I]_*,B_{jj}]=0$, 可得

$$\begin{aligned}
\Phi([[\mathrm{i}P_k,I]_*,A_{jj}+B_{jj}]) &= \Phi([[\mathrm{i}P_k,I]_*,A_{jj}]) + \Phi([[\mathrm{i}P_k,I]_*,B_{jj}]) \\
&= [[\Phi(\mathrm{i}P_k),I]_*,A_{jj}+B_{jj}] + [[\mathrm{i}P_k,\Phi(I)]_*,A_{jj}+B_{jj}] \\
&\quad + [[\mathrm{i}P_k,I]_*,\Phi(A_{jj})+\Phi(B_{jj})];
\end{aligned}$$

另一方面,

$$\begin{aligned}
\Phi([[\mathrm{i}P_k,I]_*,A_{jj}+B_{jj}]) &= [[\Phi(\mathrm{i}P_k),I]_*,A_{jj}+B_{jj}] + [[\mathrm{i}P_k,\Phi(I)]_*,A_{jj}+B_{jj}] \\
&\quad + [[\mathrm{i}P_k,I]_*,\Phi(A_{jj}+B_{jj})].
\end{aligned}$$

于是

$$[[\mathrm{i}P_k,I]_*,T]=0. \tag{2.4}$$

对 (2.4) 式左乘 P_k 右乘 P_j, 左乘 P_j 右乘 P_k, 则 $T_{kj}=T_{jk}=0$.

设 $C_{jk}\in\mathscr{A}_{jk}, 1\le k\ne j\le 2$, 则由断言 3 有

$$\begin{aligned}
\Phi(2\mathrm{i}A_{jj}C_{jk}+2\mathrm{i}B_{jj}C_{jk}) &= \Phi(2\mathrm{i}A_{jj}C_{jk}) + \Phi(2\mathrm{i}B_{jj}C_{jk}) \\
&= \Phi([[\mathrm{i}P_j,A_{jj}]_*,C_{jk}]) + \Phi([[\mathrm{i}P_j,B_{jj}]_*,C_{jk}]) \\
&= [[\Phi(\mathrm{i}P_j),A_{jj}+B_{jj}]_*,C_{jk}] + [[\mathrm{i}P_j,\Phi(A_{jj})+\Phi(B_{jj})]_*,C_{jk}] \\
&\quad + [[\mathrm{i}P_j,A_{jj}+B_{jj}]_*,\Phi(C_{jk})];
\end{aligned}$$

另一方面

$$\begin{aligned}
\Phi(2\mathrm{i}A_{jj}C_{jk}+2\mathrm{i}B_{jj}C_{jk}) &= \Phi([[\mathrm{i}P_j,A_{jj}+B_{jj}]_*,C_{jk}]) \\
&= [[\Phi(\mathrm{i}P_j),A_{jj}+B_{jj}]_*,C_{jk}] + [[\mathrm{i}P_j,\Phi(A_{jj}+B_{jj})]_*,C_{jk}] \\
&\quad + [[\mathrm{i}P_j,A_{jj}+B_{jj}]_*,\Phi(C_{jk})].
\end{aligned}$$

于是 $[[\mathrm{i}P_j,T]_*,C_{jk}]=0$. 从而由 $\mathscr{A}$ 的素性, $T_{jj}=0$.

再由断言 3 有

$$\begin{aligned}\Phi(C_{kj}A_{jj}+C_{kj}B_{jj})&=\Phi(C_{kj}A_{jj})+\Phi(C_{kj}B_{jj})\\&=\Phi([[P_k,C_{kj}]_*,A_{jj}])+\Phi([[P_k,C_{kj}]_*,B_{jj}])\\&=[[\Phi(P_k),C_{kj}]_*,A_{jj}+B_{jj}]+[[P_k,\Phi(C_{kj})]_*,A_{jj}+B_{jj}]\\&\quad+[[P_k,C_{kj}]_*,\Phi(A_{jj})+\Phi(B_{jj})];\end{aligned}$$

另一方面, 有

$$\begin{aligned}\Phi(C_{kj}A_{jj}+C_{kj}B_{jj})&=\Phi([[P_k,C_{kj}]_*,A_{jj}+B_{jj}])\\&=[[\Phi(P_k),C_{kj}]_*,A_{jj}+B_{jj}]+[[P_k,\Phi(C_{kj})]_*,A_{jj}+B_{jj}]\\&\quad+[[P_k,C_{kj}]_*,\Phi(A_{jj}+B_{jj})].\end{aligned}$$

于是 $[[P_k,C_{kj}]_*,T]=0$. 由 $\mathscr{A}$ 的素性, 从而 $T_{kk}=0$.

断言 5 对任意的 $A_{11}\in\mathscr{A}_{11}$, $B_{12}\in\mathscr{A}_{12}$, $C_{22}\in\mathscr{A}_{22}$, 有

$$\Phi(A_{11}+B_{12}+C_{22})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{22}).$$

设 $T=\Phi(A_{11}+B_{12}+C_{22})-\Phi(A_{11})-\Phi(B_{12})-\Phi(C_{22})$. 由于 $\Phi(0)=0$ 且 $[[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)]_*,A_{11}]=[[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)]_*,C_{22}]=0$, 则一方面

$$\begin{aligned}&\Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,A_{11}+B_{12}+C_{22}\right]\right)\\&=\Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,A_{11}\right]\right)+\Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,B_{12}\right]\right)+\Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,C_{22}\right]\right)\\&=\left[\left[\Phi\left(\frac{\mathrm{i}}{2}P_1\right),\mathrm{i}(P_2-P_1)\right]_*,A_{11}+B_{12}+C_{22}\right]+\left[\left[\frac{\mathrm{i}}{2}P_1,\Phi(\mathrm{i}(P_2-P_1))\right]_*,A_{11}+B_{12}+C_{22}\right]\\&\quad+\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{22})\right];\end{aligned}$$

另一方面

$$\begin{aligned}&\Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,A_{11}+B_{12}+C_{22}\right]\right)\\&=\left[\left[\Phi\left(\frac{\mathrm{i}}{2}P_1\right),\mathrm{i}(P_2-P_1)\right]_*,A_{11}+B_{12}+C_{22}\right]+\left[\left[\frac{\mathrm{i}}{2}P_1,\Phi(\mathrm{i}(P_2-P_1))\right]_*,A_{11}+B_{12}+C_{22}\right]\\&\quad+\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,\Phi(A_{11}+B_{12}+C_{22})\right].\end{aligned}$$

于是

$$\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,T\right]=0. \tag{2.5}$$

对 (2.5) 式左乘 P_1 右乘 P_2, 左乘 P_2 右乘 P_1, 可得 $T_{12}=T_{21}=0$.

设 $X_{12}\in\mathscr{A}_{12}$, 由 $\Phi(0)=0$ 和 $[[\mathrm{i}P_1,C_{22}]_*,X_{12}]=[[\mathrm{i}P_1,B_{12}]_*,X_{12}]=0$, 可得

$$\begin{aligned}&\Phi([[\mathrm{i}P_1,A_{11}+B_{12}+C_{22}]_*,X_{12}])\\&=\Phi([[\mathrm{i}P_1,C_{22}]_*,X_{12}])+\Phi([[\mathrm{i}P_1,B_{12}]_*,X_{12}])+\Phi([[\mathrm{i}P_1,A_{11}]_*,X_{12}])\end{aligned}$$

$$= [[\Phi(\mathrm{i}P_1), A_{11} + B_{12} + C_{22}]_*, X_{12}] + [[\mathrm{i}P_1, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{22})]_*, X_{12}] + [[\mathrm{i}P_1, A_{11} + B_{12} + C_{22}]_*, \Phi(X_{12})];$$

另一方面, 有

$$\begin{aligned}&\Phi([[\mathrm{i}P_1, A_{11} + B_{12} + C_{22}]_*, X_{12}])\\ &= [[\Phi(\mathrm{i}P_1), A_{11} + B_{12} + C_{22}]_*, X_{12}] + [[\mathrm{i}P_1, \Phi(A_{11} + B_{12} + C_{22})]_*, X_{12}]\\ &\quad + [[\mathrm{i}P_1, A_{11} + B_{12} + C_{22}]_*, \Phi(X_{12})].\end{aligned}$$

于是 $[[\mathrm{i}P_1, T]_*, X_{12}] = 0$. 从而由 $\mathscr{A}$ 的素性, $T_{11} = 0$.

由断言 3 得

$$\begin{aligned}&\Phi(X_{12}C_{22} - A_{11}X_{12}) + \Phi([[P_1, X_{12}]_*, B_{12}])\\ &= \Phi([[P_1, X_{12}]_*, A_{11}]) + \Phi([[P_1, X_{12}]_*, C_{22}]) + \Phi([[P_1, X_{12}]_*, B_{12}])\\ &= [[\Phi(P_1), X_{12}]_*, A_{11} + B_{12} + C_{22}] + [[P_1, \Phi(X_{12})]_*, A_{11} + B_{12} + C_{22}]\\ &\quad + [[P_1, X_{12}]_*, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{22})];\end{aligned}$$

另一方面, 因为 $[[P_1, X_{12}]_*, B_{12}] = 0$, 所以

$$\begin{aligned}&\Phi(X_{12}C_{22} - A_{11}X_{12}) + \Phi([[P_1, X_{12}]_*, B_{12}])\\ &= \Phi([[P_1, X_{12}]_*, A_{11} + C_{22}]) + \Phi([[P_1, X_{12}]_*, B_{12}])\\ &= \Phi([[P_1, X_{12}]_*, A_{11} + B_{12} + C_{22}])\\ &= [[\Phi(P_1), X_{12}]_*, A_{11} + B_{12} + C_{22}] + [[P_1, \Phi(X_{12})]_*, A_{11} + B_{12} + C_{22}]\\ &\quad + [[P_1, X_{12}]_*, \Phi(A_{11} + B_{12} + C_{22})].\end{aligned}$$

于是 $[[P_1, X_{12}]_*, T] = 0$. 由 $\mathscr{A}$ 的素性, 从而 $T_{22} = 0$.

断言 6 对任意的 $A_{11} \in \mathscr{A}_{11}, B_{12} \in \mathscr{A}_{12}, C_{21} \in \mathscr{A}_{21}, D_{22} \in \mathscr{A}_{22}$, 有

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

设 $T = \Phi(A_{11} + B_{12} + C_{21} + D_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21}) - \Phi(D_{22})$. 由断言 1 和 $[[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2 - P_1)]_*, A_{11}] = [[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2 - P_1)]_*, D_{22}] = 0$, 有

$$\begin{aligned}\Phi(B_{12} - C_{21}) &= \Phi(B_{12}) + \Phi(-C_{21})\\ &= \Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2 - P_1)\right]_*, A_{11}\right]\right) + \Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2 - P_1)\right]_*, B_{12}\right]\right)\\ &\quad + \Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2 - P_1)\right]_*, C_{21}\right]\right) + \Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2 - P_1)\right]_*, D_{22}\right]\right)\\ &= \left[\left[\Phi\left(\frac{\mathrm{i}}{2}P_1\right), \mathrm{i}(P_2 - P_1)\right]_*, A_{11} + B_{12} + C_{21} + D_{22}\right]\\ &\quad + \left[\left[\frac{\mathrm{i}}{2}P_1, \Phi(\mathrm{i}(P_2 - P_1))\right]_*, A_{11} + B_{12} + C_{21} + D_{22}\right]\\ &\quad + \left[\left[\frac{\mathrm{i}}{2}P_1, \mathrm{i}(P_2 - P_1)\right]_*, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})\right];\end{aligned}$$

另一方面

$$\begin{aligned}\Phi(B_{12}-C_{21})&=\Phi\left(\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,A_{11}+B_{12}+C_{21}+D_{22}\right]\right)\\&=\left[\left[\Phi\left(\frac{\mathrm{i}}{2}P_1\right),\mathrm{i}(P_2-P_1)\right]_*,A_{11}+B_{12}+C_{21}+D_{22}\right]\\&\quad+\left[\left[\frac{\mathrm{i}}{2}P_1,\Phi(\mathrm{i}(P_2-P_1))\right]_*,A_{11}+B_{12}+C_{21}+D_{22}\right]\\&\quad+\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,\Phi(A_{11}+B_{12}+C_{21}+D_{22})\right].\end{aligned}$$

于是

$$\left[\left[\frac{\mathrm{i}}{2}P_1,\mathrm{i}(P_2-P_1)\right]_*,T\right]=0. \tag{2.6}$$

对 (2.6) 式左乘 P_1 右乘 P_2, 左乘 P_2 右乘 P_1, 则 $T_{12}=T_{21}=0$. 类似于断言 5 的证明过程, 可得 $T_{11}=T_{22}=0$.

最后, 由断言 3, 4 和 6, 则对任意的 $A,B\in\mathscr{A}$, $\Phi(A+B)=\sum_{j,k=1}^2\Phi(A_{jk}+B_{jk})=\sum_{j,k=1}^2\Phi(A_{jk})+\sum_{j,k=1}^2\Phi(B_{jk})=\Phi(A)+\Phi(B)$, 即 Φ 是可加的. 证毕.

引理 2.6 $\Phi(\mathscr{A}_{jk})\subseteq\mathscr{A}_{jk}$ 且 $\Phi(P_j)=0$, $1\le j\ne k\le 2$.

证明 对 (2.1) 式左乘 P_j 右乘 P_k, 左乘 P_k 右乘 P_j, 则 $P_j\Phi(P_j)P_k=P_k\Phi(P_j)P_j=0$. 从而

$$\Phi(P_j)=P_j\Phi(P_j)P_j+P_k\Phi(P_j)P_k. \tag{2.7}$$

设 $A_{jk}\in\mathscr{A}_{jk}$, 由 $[[A_{jk},P_j]_*,P_j]=0$ 和 $\Phi(0)=0$, 得

$$\begin{aligned}0&=\Phi([[A_{jk},P_j]_*,P_j])=[[\Phi(A_{jk}),P_j]_*,P_j]+[[A_{jk},\Phi(P_j)]_*,P_j]\\&=\Phi(A_{jk})P_j-P_j\Phi(A_{jk})^*P_j-P_j\Phi(A_{jk})P_j+P_j\Phi(A_{jk})^*\\&\quad-\Phi(P_j)A_{jk}^*-A_{jk}\Phi(P_j)+P_j\Phi(P_j)A_{jk}^*.\end{aligned}$$

对上式左乘 P_j 右乘 P_k, 左乘 P_k 右乘 P_j, 有

$$A_{jk}\Phi(P_j)P_k=P_j\Phi(A_{jk})^*P_k,\quad P_k\Phi(P_j)A_{jk}^*=P_k\Phi(A_{jk})P_j. \tag{2.8}$$

因此, $P_k\Phi(P_j)A_{jk}^*=P_k\Phi(P_j)^*A_{jk}^*$. 从而由 $\mathscr{A}$ 的素性,

$$P_k\Phi(P_j)P_k=P_k\Phi(P_j)^*P_k. \tag{2.9}$$

由引理 2.5,

$$\begin{aligned}-\Phi(A_{jk})&=\Phi(-A_{jk})=\Phi([[P_j,A_{jk}]_*,P_j])\\&=[[\Phi(P_j),A_{jk}]_*,P_j]+[[P_j,\Phi(A_{jk})]_*,P_j]+[[P_j,A_{jk}]_*,\Phi(P_j)]\\&=-P_j\Phi(P_j)A_{jk}+A_{jk}\Phi(P_j)^*+2P_j\Phi(A_{jk})P_j-\Phi(A_{jk})P_j\\&\quad-P_j\Phi(A_{jk})+A_{jk}\Phi(P_j)-\Phi(P_j)A_{jk}.\end{aligned}$$

对上式两边分别同乘 P_j, P_k 以及左乘 P_j 右乘 P_k, 则

$$P_j\Phi(A_{jk})P_j=P_k\Phi(A_{jk})P_k=0 \tag{2.10}$$

以及

$$A_{jk}\Phi(P_j)^*P_k+A_{jk}\Phi(P_j)P_k=2P_j\Phi(P_j)A_{jk}. \tag{2.11}$$

由 (2.9) 和 (2.11) 式, 则 $A_{jk}\Phi(P_j)P_k = P_j\Phi(P_j)A_{jk}$. 从而由引理 2.3 和 (2.7) 式, 有

$$\Phi(P_j) = P_j\Phi(P_j)P_j + P_k\Phi(P_j)P_k \in \mathbb{C}I. \tag{2.12}$$

对任意的 $B_{jk} \in \mathscr{A}_{jk}$, 由 $[[P_j, A_{jk}]_*, B_{jk}] = 0$, 有

$$\begin{aligned} 0 &= \Phi([[P_j, A_{jk}]_*, B_{jk}]) \\ &= [[\Phi(P_j), A_{jk}]_*, B_{jk}] + [[P_j, \Phi(A_{jk})]_*, B_{jk}] + [[P_j, A_{jk}]_*, \Phi(B_{jk})] \\ &= -\Phi(A_{jk})B_{jk} + B_{jk}\Phi(A_{jk})P_j + A_{jk}\Phi(B_{jk}) - \Phi(B_{jk})A_{jk}. \end{aligned}$$

对上式两边分别同乘 P_j 和 P_k, 则

$$B_{jk}\Phi(A_{jk})P_j = -A_{jk}\Phi(B_{jk})P_j, \quad P_k\Phi(B_{jk})A_{jk} = -P_k\Phi(A_{jk})B_{jk}. \tag{2.13}$$

特别地, 有

$$P_k\Phi(A_{jk})A_{jk} = A_{jk}\Phi(A_{jk})P_j = 0. \tag{2.14}$$

由 (2.13) 式, 则对任意的 $B_{jk}, X_{jk} \in \mathscr{A}_{jk}$,

$$B_{jk}\Phi(A_{jk})X_{jk} = A_{jk}\Phi(X_{jk})B_{jk}. \tag{2.15}$$

由引理 2.3 和 (2.15) 式, 从而 $P_k\Phi(A_{jk})X_{jk} + A_{jk}\Phi(X_{jk})P_j \in \mathbb{C}I$. 由此可得

$$P_k\Phi(A_{jk})X_{jk} \in \mathbb{C}P_k \quad 且 \quad A_{jk}\Phi(X_{jk})P_j \in \mathbb{C}P_j.$$

从而存在映射 $F, G : \mathscr{A}_{jk} \times \mathscr{A}_{jk} \to \mathbb{C}$, 使得

$$F(A_{jk}, X_{jk})P_k = P_k\Phi(A_{jk})X_{jk} \tag{2.16}$$

且

$$G(A_{jk}, X_{jk})P_j = X_{jk}\Phi(A_{jk})P_j. \tag{2.17}$$

由 (2.16) 和 (2.17) 式, 则对任意的 $A_{jk}, X_{jk}, Y_{jk} \in \mathscr{A}_{jk}$,

$$F(A_{jk}, X_{jk})Y_{jk} = Y_{jk}\Phi(A_{jk})X_{jk} = G(A_{jk}, Y_{jk})X_{jk}. \tag{2.18}$$

假设存在 $(A_0, V_0) \in \mathscr{A}_{jk} \times \mathscr{A}_{jk}$ 使得 $F(A_0, V_0) \neq 0$, 则由 (2.18) 式, 对任意的 $Y_{jk} \in \mathscr{A}_{jk}$, 有

$$F(A_0, V_0)Y_{jk} = G(A_0, Y_{jk})V_0.$$

从而 $Y_{jk} = \lambda(Y_{jk})V_0$, 其中 $\lambda(Y_{jk}) = \frac{G(A_0, Y_{jk})}{F(A_0, V_0)} \in \mathbb{C}$. 在 (2.16) 中取 $X_{jk} = V_0$, 则 $F(A_{jk}, V_0)P_k = P_k\Phi(A_{jk})V_0$. 于是由 (2.14) 式, 对任意的 $A_{jk} \in \mathscr{A}_{jk}$, 有

$$F(A_{jk}, V_0)\lambda(A_{jk})P_k = P_k\Phi(A_{jk})\lambda(A_{jk})V_0 = P_k\Phi(A_{jk})A_{jk} = 0.$$

从而 $\lambda(A_{jk}) = 0$ 或 $F(A_{jk}, V_0) = 0$. 如果 $\lambda(A_{jk}) = 0$, 则 $A_{jk} = \lambda(A_{jk})V_0 = 0$, 进而 $F(A_{jk}, V_0) = 0$. 于是对任意的 $A_{jk} \in \mathscr{A}_{jk}$, 总有 $F(A_{jk}, V_0) = 0$. 特别地, $F(A_0, V_0) = 0$, 矛盾. 这说明对任意的 $A_{jk}, X_{jk} \in \mathscr{A}_{jk}$, 都有 $F(A_{jk}, X_{jk}) = 0$. 由 (2.16) 式, 从而 $P_k\Phi(A_{jk})X_{jk} = 0$. 由 $\mathscr{A}$ 的素性, 则 $P_k\Phi(A_{jk})P_j = 0$. 从而由 (2.10) 式, $\Phi(A_{jk}) = P_j\Phi(A_{jk})P_k \in \mathscr{A}_{jk}$. 结合 (2.8) 式, 则对任意的 $A_{jk} \in \mathscr{A}_{jk}$, 有 $A_{jk}\Phi(P_j)P_k = 0$. 于是 $P_k\Phi(P_j)P_k = 0$, 进而由 (2.12), $\Phi(P_j) = 0$, $j = 1, 2$. 证毕.

引理 2.7 存在可加映射 $f_j : \mathscr{A}_{jj} \to \mathbb{C}I$, 使得对任意的 $A_{jj} \in \mathscr{A}_{jj}$, 有 $\Phi(A_{jj}) - f_j(A_{jj}) \in \mathscr{A}_{jj}$ $(j = 1, 2)$.

证明 设 $A_{jj}\in\mathscr{A}_{jj}$ 且 $k\neq j$, 则由引理 2.6 有

$$0=\Phi([[P_k,A_{jj}]_*,P_j])=[[P_k,\Phi(A_{jj})]_*,P_j]=P_k\Phi(A_{jj})P_j+P_j\Phi(A_{jj})P_k. \tag{2.19}$$

对 (2.19) 式左乘 P_j 右乘 P_k 和左乘 P_k 右乘 P_j, 则 $P_j\Phi(A_{jj})P_k=P_k\Phi(A_{jj})P_j=0$. 从而

$$\Phi(A_{jj})=P_1\Phi(A_{jj})P_1+P_2\Phi(A_{jj})P_2. \tag{2.20}$$

对任意的 $B_{kk}\in\mathscr{A}_{kk}$, 由 $[A_{jj},B_{kk}]_*=0$ 可得 $[[\Phi(A_{jj}),B_{kk}]_*+[A_{jj},\Phi(B_{kk})]_*,C]=0$ 对任意的 $C\in\mathscr{A}$ 成立. 从而由 $\mathscr{A}$ 是因子可知

$$[\Phi(A_{jj}),B_{kk}]_*+[A_{jj},\Phi(B_{kk})]_*\in\mathbb{C}I.$$

也就是

$$[P_2\Phi(A_{11})P_2,B_{22}]_*+[A_{11},P_1\Phi(B_{22})P_1]_*\in\mathbb{C}I \tag{2.21}$$

和

$$[P_1\Phi(A_{22})P_1,B_{11}]_*+[A_{22},P_2\Phi(B_{11})P_2]_*\in\mathbb{C}I. \tag{2.22}$$

由 (2.21) 知 $[P_2\Phi(A_{11})P_2,B_{22}]_*\in\mathbb{C}P_2$ 对任意的 $B_{22}\in\mathscr{A}_{22}$ 成立. 由引理 2.2, 则 $P_2\Phi(A_{11})P_2\in\mathbb{C}P_2$. 从而存在映射 $f_1:\mathscr{A}_{11}\to\mathbb{C}I$, 使得 $P_2\Phi(A_{11})P_2=f_1(A_{11})P_2$. 类似地, 由 (2.22) 式可知存在映射 $f_2:\mathscr{A}_{22}\to\mathbb{C}I$, 使得 $P_1\Phi(A_{22})P_1=f_2(A_{22})P_1$. 由 (2.20) 式, 因此 $\Phi(A_{jj})-f_j(A_{jj})=P_j\Phi(A_{jj})P_j-f_j(A_{jj})P_j\in\mathscr{A}_{jj}$. 证毕.

注 2.2 由引理 2.7, 分别定义映射 $f:\mathscr{A}\to\mathbb{C}I$ 和 $\Delta:\mathscr{A}\to\mathscr{A}$ 为 $f(A)=f_1(P_1AP_1)+f_2(P_2AP_2)$ 和 $\Delta(A)=\Phi(A)-f(A)$, 则由引理 2.5–2.7, Δ 具有下列性质:

(a) Δ 是可加的;

(b) $\Delta(\mathscr{A}_{jk})\subseteq\mathscr{A}_{jk}$, $j,k=1,2$;

(c) $\Delta|_{\mathscr{A}_{jk}}=\Phi|_{\mathscr{A}_{jk}}$, $1\leq j\neq k\leq 2$.

引理 2.8 Δ 是可加的导子.

证明 设 $1\leq j\neq k\leq 2$, $A_{jj},B_{jj}\in\mathscr{A}_{jj}$, $A_{jk},B_{jk}\in\mathscr{A}_{jk}$. 通过以下几个断言证明.

断言 1 (a) $\Delta(A_{jj}B_{jk})=\Delta(A_{jj})B_{jk}+A_{jj}\Delta(B_{jk})$;

(b) $\Delta(A_{jk}B_{kk})=\Delta(A_{jk})B_{kk}+A_{jk}\Delta(B_{kk})$.

(a) 由 Δ 的性质和引理 2.6, 则

$$\begin{aligned}-\Delta(A_{jj}B_{jk})&=\Phi([[P_j,B_{jk}]_*,A_{jj}])=[[P_j,\Delta(B_{jk})]_*,A_{jj}]+[[P_j,B_{jk}]_*,\Delta(A_{jj})+f(A_{jj})]\\&=-A_{jj}\Delta(B_{jk})-\Delta(A_{jj})B_{jk},\end{aligned}$$

即 $\Delta(A_{jj}B_{jk})=\Delta(A_{jj})B_{jk}+A_{jj}\Delta(B_{jk})$. 类似地, (b) 也成立.

断言 2 $\Delta(A_{jj}B_{jj})=\Delta(A_{jj})B_{jj}+A_{jj}\Delta(B_{jj})$.

由断言 1 (a), 对任意的 $X_{jk}\in\mathscr{A}_{jk}$, 有

$$\begin{aligned}\Delta(A_{jj}B_{jj})X_{jk}+A_{jj}B_{jj}\Delta(X_{jk})&=\Delta(A_{jj}B_{jj}X_{jk})=\Delta(A_{jj})B_{jj}X_{jk}+A_{jj}\Delta(B_{jj}X_{jk})\\&=\Delta(A_{jj})B_{jj}X_{jk}+A_{jj}\Delta(B_{jj})X_{jk}+A_{jj}B_{jj}\Delta(X_{jk}).\end{aligned}$$

从而

$$(\Delta(A_{jj}B_{jj})-\Delta(A_{jj})B_{jj}-A_{jj}\Delta(B_{jj}))X_{jk}=0.$$

由 $\mathscr{A}$ 的素性, 则 $\Delta(A_{jj}B_{jj})=\Delta(A_{jj})B_{jj}+A_{jj}\Delta(B_{jj})$.

断言 3　$\Delta(A_{jk}B_{kj}) = \Delta(A_{jk})B_{kj} + A_{jk}\Delta(B_{kj})$.

由 Δ 的性质 (b), (c) 及断言 1(a), 则

$$\begin{aligned}\Delta(A_{jk}B_{kj})X_{jk} + A_{jk}B_{kj}\Delta(X_{jk}) &= \Delta(A_{jk}B_{kj}X_{jk}) = \Phi([[A_{jk}, B_{kj}]_*, X_{jk}]) \\ &= [[\Delta(A_{jk}), B_{kj}]_*, X_{jk}] + [[A_{jk}, \Delta(B_{kj})]_*, X_{jk}] \\ &\quad + [[A_{jk}, B_{kj}]_*, \Delta(X_{jk})] \\ &= \Delta(A_{jk})B_{kj}X_{jk} + A_{jk}\Delta(B_{kj})X_{jk} + A_{jk}B_{kj}\Delta(X_{jk}).\end{aligned}$$

从而对任意的 $X_{jk} \in \mathscr{A}_{jk}$, 有 $(\Delta(A_{jk}B_{kj}) - \Delta(A_{jk})B_{kj} - A_{jk}\Delta(B_{kj}))X_{jk} = 0$. 由 $\mathscr{A}$ 的素性, 于是 $\Delta(A_{jk}B_{kj}) = \Delta(A_{jk})B_{kj} + A_{jk}\Delta(B_{kj})$.

最后, 由断言 1–3, 则 $\Delta(AB) = \Delta(A)B + A\Delta(B)$ 对任意的 $A, B \in \mathscr{A}$ 成立. 从而 Δ 是可加的导子. 证毕.

引理 2.9　对任意的 $A \in \mathscr{A}$, 有 $\Delta(A^*) = \Delta(A)^*$.

证明　设 $1 \le j \ne k \le 2$ 且 $A_{jk} \in \mathscr{A}_{jk}$. 由 Δ 的性质 (b), (c) 和引理 2.6, 有

$$\Delta(A_{jk}) + \Delta(A_{jk}^*) = \Phi([[A_{jk}, P_k]_*, P_k]) = [[\Delta(A_{jk}), P_k]_*, P_k] = \Delta(A_{jk}) + \Delta(A_{jk})^*.$$

从而

$$\Delta(A_{jk}^*) = \Delta(A_{jk})^*. \tag{2.23}$$

设 $A_{jj} \in \mathscr{A}_{jj}$. 由 Δ 的性质 (b), (c) 和引理 2.6, 对任意的 $X_{jk} \in \mathscr{A}_{jk}$, 有

$$\begin{aligned}\Delta(A_{jj}X_{jk}) &= \Phi([[A_{jj}, X_{jk}]_*, P_k]) = [[\Delta(A_{jj}) + f(A_{jj}), X_{jk}]_*, P_k] + [[A_{jj}, \Delta(X_{jk})]_*, P_k] \\ &= \Delta(A_{jj})X_{jk} + A_{jj}\Delta(X_{jk}) + (f(A_{jj}) - f(A_{jj})^*)X_{jk}.\end{aligned}$$

从而, 由引理 2.8 的断言 1 (a) 以及 $f(A_{jj})^* = f(A_{jj})$, 即 $f(A_{jj}) \in \mathbb{R}I$. 再由 Δ 的性质 (b), (c) 和引理 2.6 知

$$\begin{aligned}\Phi([[A_{jj}, P_j]_*, A_{jk}]) &= [[\Delta(A_{jj}) + f(A_{jj}), P_j]_*, A_{jk}] + [[A_{jj}, P_j]_*, \Delta(A_{jk})] \\ &= \Delta(A_{jj})A_{jk} - \Delta(A_{jj})^*A_{jk} + A_{jj}\Delta(A_{jk}) - A_{jj}^*\Delta(A_{jk}).\end{aligned}$$

另一方面, 由 Δ 的性质 (c) 及引理 2.8 的断言 1 (a),

$$\begin{aligned}\Phi([[A_{jj}, P_j]_*, A_{jk}]) &= \Delta(A_{jj}A_{jk}) - \Delta(A_{jj}^*A_{jk}) \\ &= \Delta(A_{jj})A_{jk} + A_{jj}\Delta(A_{jk}) - \Delta(A_{jj}^*)A_{jk} - A_{jj}^*\Delta(A_{jk}).\end{aligned}$$

从而 $(\Delta(A_{jj}^*) - \Delta(A_{jj})^*)A_{jk} = 0$. 由 $\mathscr{A}$ 的素性, 则 $\Delta(A_{jj}^*) = \Delta(A_{jj})^*$, $j = 1, 2$. 结合 (2.23) 式, 于是 $\Delta(A^*) = \Delta(A)^*$ 对任意的 $A \in \mathscr{A}$ 成立. 证毕.

定理 2.1 的证明　由引理 2.8 和 2.9, 则 Δ 是可加的 $*$- 导子. 从而

$$\Delta([[A, B]_*, C]) = [[\Delta(A), B]_*, C] + [[A, \Delta(B)]_*, C] + [[A, B]_*, \Delta(C)]$$

对所有的 $A, B, C \in \mathscr{A}$ 均成立. 由引理 2.9 证明过程中的结论 $f(\mathscr{A}_{jj}) \subseteq \mathbb{R}I$ $(j = 1, 2)$ 以及 f 的定义可知, 对任意的 $A \in \mathscr{A}$, 有 $f(A) \in \mathbb{R}I$. 进而由 Δ 的定义

$$\begin{aligned}f([[A, B]_*, C]) &= \Phi([[A, B]_*, C]) - \Delta([[A, B]_*, C]) \\ &= [[\Delta(A) + f(A), B]_*, C] + [[A, \Delta(B) + f(B)]_*, C] \\ &\quad + [[A, B]_*, \Delta(C) + f(C)] - \Delta([[A, B]_*, C])\end{aligned}$$

$$
\begin{aligned}
&= [[\Delta(A), B]_*, C] + [[A, \Delta(B)]_*, C] + [[A, B]_*, \Delta(C)] \\
&\quad - \Delta([[A, B]_*, C]) + [[A, f(B)]_*, C] \\
&= [[A, f(B)]_*, C].
\end{aligned}
$$

在上式中取 $A = iI$, 并用 $\frac{i}{2}C$ 替换 C, 则对任意的 $B, C \in \mathscr{A}$, 有 $f([B, C]) = 0$. 因此

$$f(B)[A - A^*, C] = [[A, f(B)]_*, C] = f([[A, B]_*, C]) = 0 \tag{2.24}$$

对所有的 $A, B, C \in \mathscr{A}$ 成立. 在 (2.24) 式中, 用 iA 代替 A 可得

$$f(B)[A + A^*, C] = 0. \tag{2.25}$$

结合 (2.24) 和 (2.25) 式, 则 $f(B)[A, C] = 0$. 由 $\mathscr{A}$ 为因子且 $\dim(\mathscr{A}) > 1$ 知, 存在 $A, C \in \mathscr{A}$ 使得 $[A, C] \neq 0$. 从而对任意的 $B \in \mathscr{A}$, 有 $f(B) = 0$. 进而 $\Phi = \Delta$ 为可加的 $*$- 导子. 由 Φ 的定义, 对任意的 $A \in \mathscr{A}$, $\delta(A) = \Phi(A) + [A, T]$, 其中 $T = P_1\delta(P_1)P_2 + P_2\delta(P_2)P_1 = -T^*$. 显然, 映射 $A \to [A, T]$ 是 $\mathscr{A}$ 上的可加 $*$- 导子. 因此, δ 是可加的 $*$- 导子. 证毕.

作为定理 2.1 的应用, 我们得到以下推论.

推论 2.1 设 $\mathscr{H}$ 是无限维的复 Hilbert 空间, $\delta : B(\mathscr{H}) \to B(\mathscr{H})$ 是非线性混合 Lie 三重可导映射, 则存在 $T \in B(\mathscr{H})$ 且 $T + T^* = 0$, 使得对任意的 $A \in B(\mathscr{H})$, 有 $\delta(A) = AT - TA$.

证明 由定理 2.1 知 δ 是可加的 $*$- 导子. 由文 [8] 的结论知 δ 是线性的, 从而 δ 是内导子. 因此, 存在 $S \in B(\mathscr{H})$, 使得对任意的 $A \in B(\mathscr{H})$, 有 $\delta(A) = AS - SA$. 由于

$$A^*S - SA^* = \delta(A^*) = \delta(A)^* = S^*A^* - A^*S^*,$$

则存在 $\lambda \in \mathbb{R}$, 使得 $S + S^* = \lambda I$. 令 $T = S - \frac{1}{2}\lambda I$, 则 $T + T^* = 0$, 并且对任意的 $A \in B(\mathscr{H})$, 有 $\delta(A) = AT - TA$. 证毕.

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