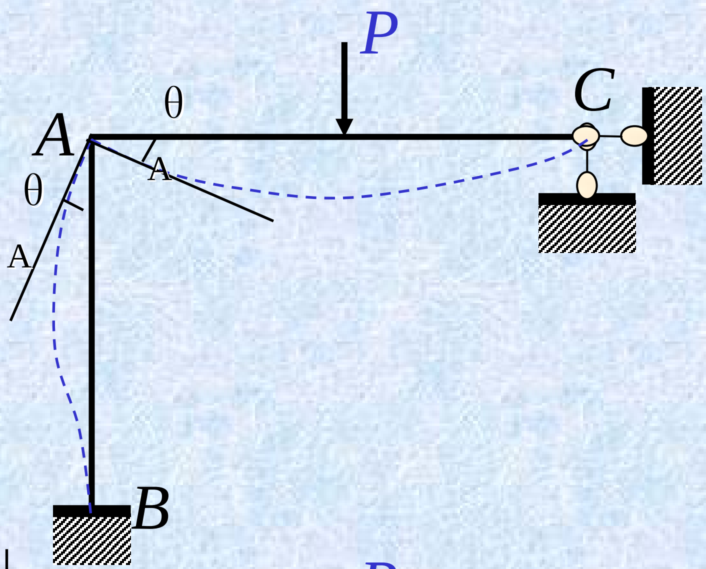
第十一章

位 移 法

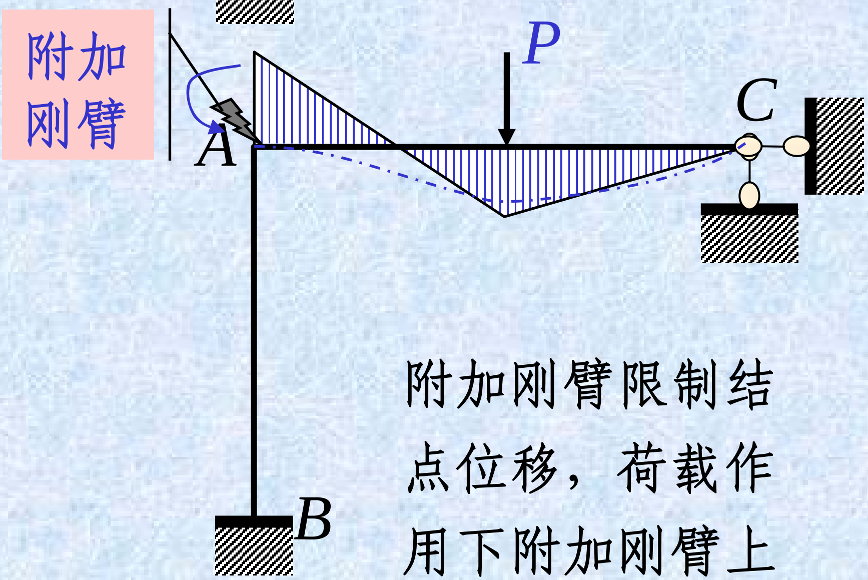
§11-1　位移法的基本概念



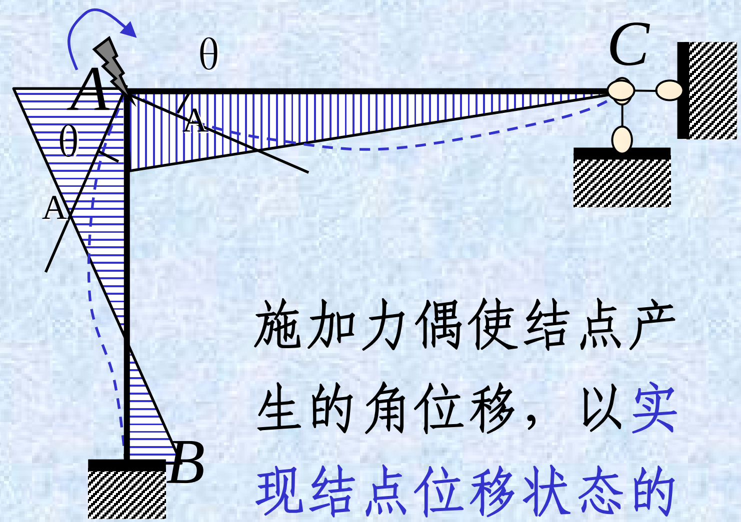
荷载效应包括:

内力效应：***M*、*Q*、*N*;**

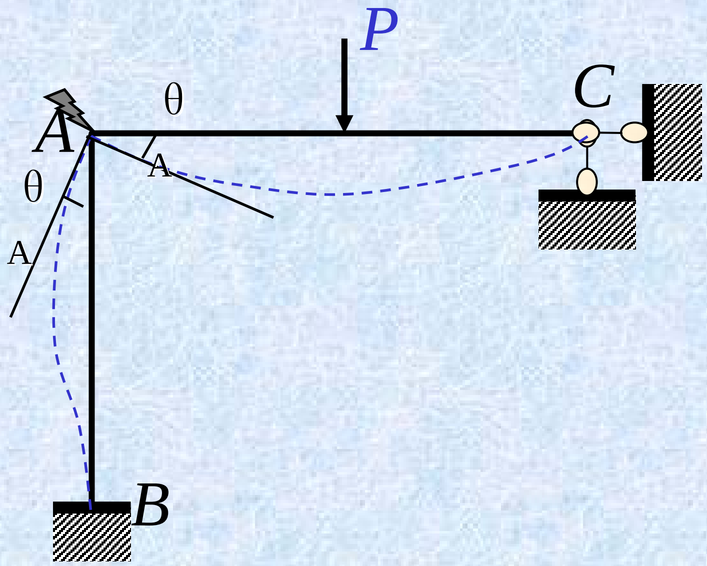
位移效应：　θ_A



附加刚臂限制结点位移，荷载作用下附加刚臂上产生附加力矩



施加力偶使结点产生的角位移，以实现结点位移状态的一致性。



实现位移状态可
分两步完成:

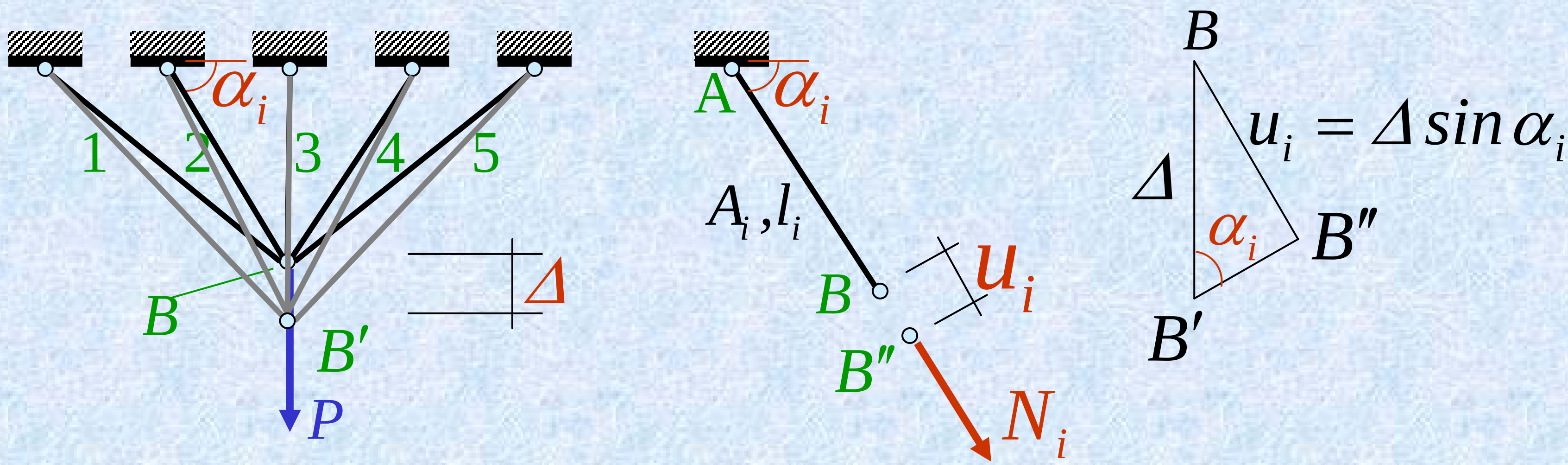
1）在可动结点上附加约束，
限制其位移，在荷载作用下，
附加约束上产生附加约束力;

2）在附加约束上施加外力，
使结构发生与原结构一致的结
点位移。

分析:

1）叠加两步作用效应，约束结构与原结构的荷载特征及
位移特征完全一致，则其内力状态也完全相等;

2）结点位移计算方法：对比两结构可发现，附加约束上
的附加内力应等于0，按此可列出基本方程。



选择基本未知量Δ

$$N_i = \frac{EA_i}{l_i} u_i$$

物理条件

$$u_i = \Delta \sin \alpha_i$$

几何条件

$$\sum N_i \sin \alpha_i = P$$

平衡条件

变形条件

$$N_i = \frac{EA_i}{l_i} \Delta \sin \alpha_i$$

$$N_i = \frac{\frac{EA_i}{l_i} \sin \alpha_i}{\sum \frac{EA_i}{l_i} \sin^2 \alpha_i} P$$

$$\sum \frac{EA_i}{l_i} \sin^2 \alpha_i \cdot \Delta = P$$

$$\Delta = \frac{P}{\sum \frac{EA_i}{l_i} \sin^2 \alpha_i}$$

位移法基本作法小结:

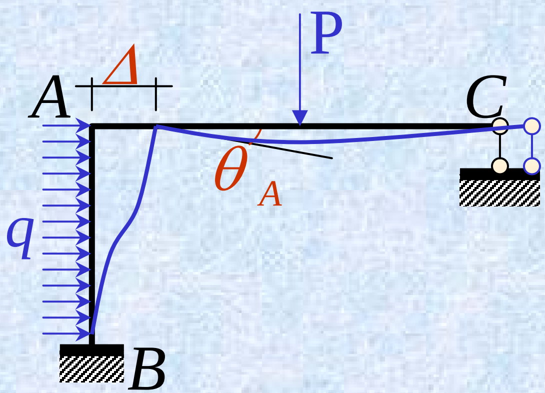
（1）基本未知量是结点位移;

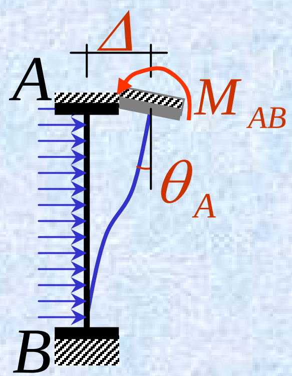
（2）基本方程的实质含义是静力平衡条件;

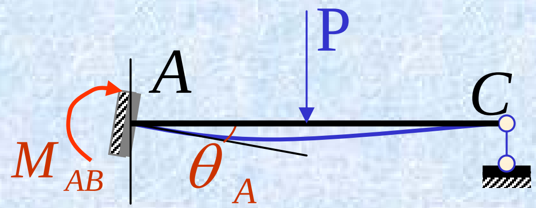
（3）建立基本方程分两步——单元分析（拆分）求得单元刚度方程，整体分析（组合）建立位移法基本方程，解方程求出基本未知量;

（4）由杆件的刚度方程求出杆件内力，画弯矩图。

关于刚架的结点未知量

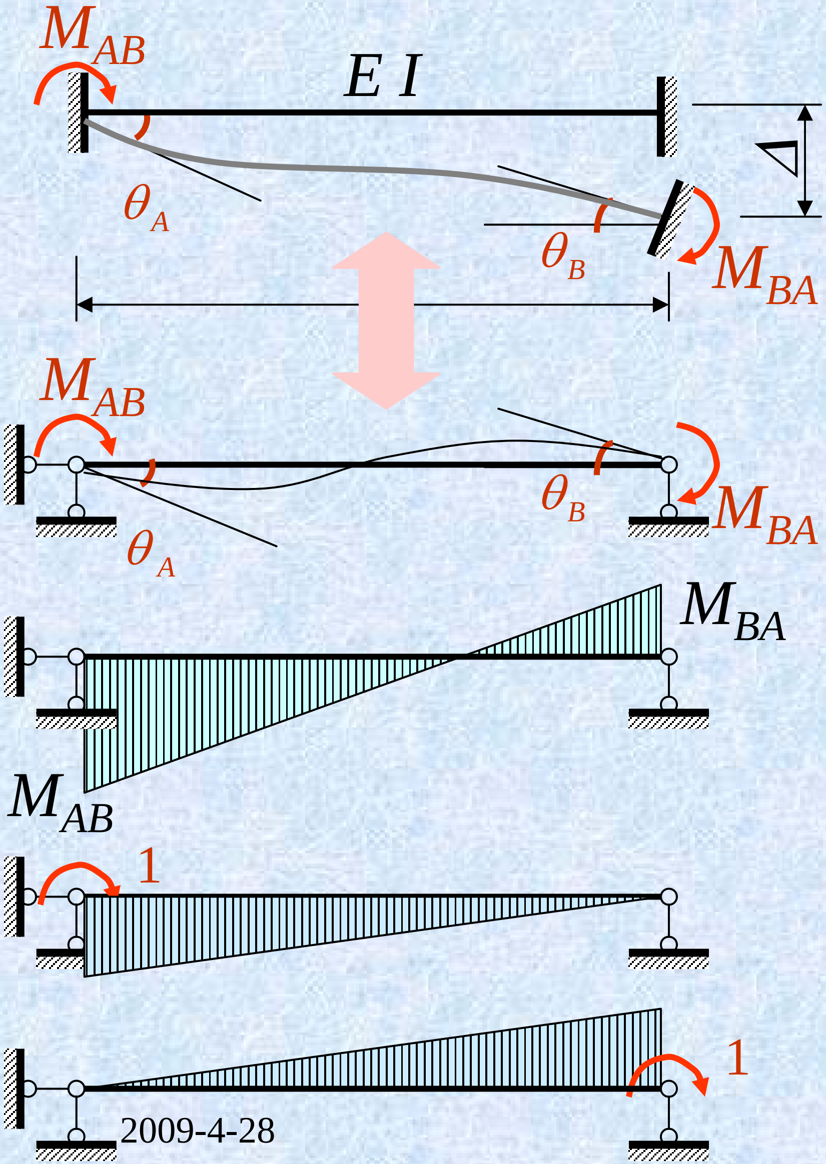






§11-2　等截面杆件的刚度方程

一、由杆端位移求杆端弯矩



♥　杆端力和杆端位移的正负规定

①杆端转角 θ_A、θ_B，弦转角 $\beta=\Delta/l$ 都以顺时针为正。

②杆端弯矩对杆端以顺时针为正，对结点或支座以逆时针为正。

（1）由杆端弯矩 M_{AB} 和 M_{BA} 引起的 θ_A 和 θ_B

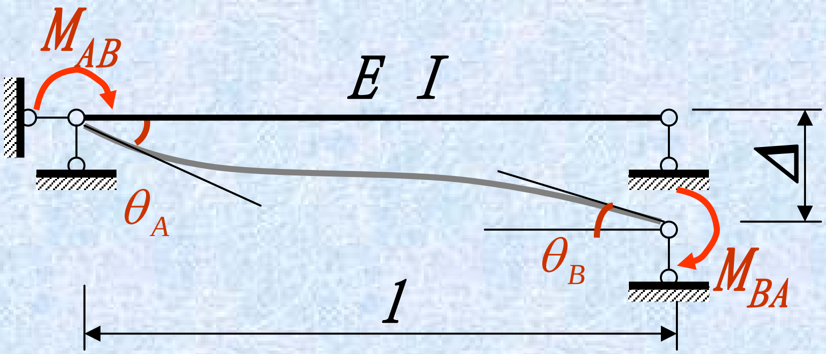
利用单位荷载法可求得

$$\theta_A=\frac{1}{EI}\left(\frac{1}{2}M_{AB}\times l\times\frac{2}{3}-M_{BA}\times l\times\frac{1}{3}\right)$$

$$=\frac{l}{EI}\left(\frac{1}{3}M_{AB}-\frac{1}{6}M_{BA}\right)$$

设 $\frac{EI}{l}=i$　　$\theta_A=\frac{1}{3i}M_{AB}-\frac{1}{6i}M_{BA}$

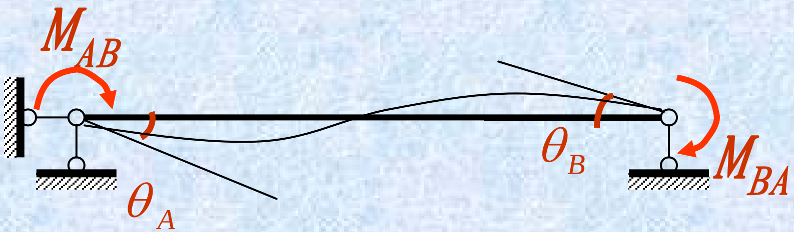
同理可得　　$\theta_B=-\frac{1}{6i}M_{AB}+\frac{1}{3i}M_{BA}$



$$\theta_A = \frac{1}{3i} M_{AB} - \frac{1}{6i} M_{BA}$$

$$\theta_B = -\frac{1}{6i} M_{AB} + \frac{1}{3i} M_{BA}$$

（2）由于相对线位移Δ引起的θ_A和θ_B

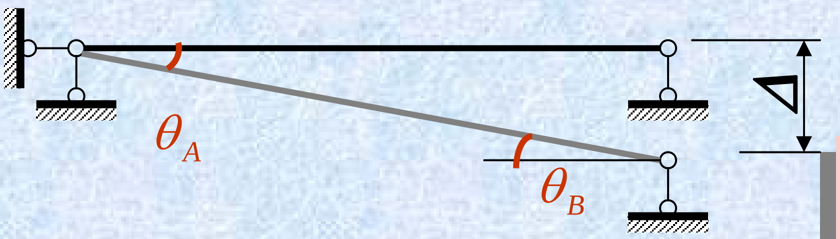


$$\theta_A = \theta_B = \frac{\Delta}{l}$$

以上两过程的叠加

$$\theta_A = \frac{1}{3i} M_{AB} - \frac{1}{6i} M_{BA} + \frac{\Delta}{l}$$

$$\theta_B = -\frac{1}{6i} M_{AB} + \frac{1}{3i} M_{BA} + \frac{\Delta}{l}$$



我们的任务是要由杆端位移求杆端力，变换上面的式子可得：

$$\left.\begin{aligned} M_{AB} &= 4i\theta_A + 2i\theta_B - 6i\frac{\Delta}{l} \\ M_{BA} &= 2i\theta_A + 4i\theta_B - 6i\frac{\Delta}{l} \end{aligned}\right\} \cdots\cdots(1)$$

$$Q_{AB} = Q_{BA} = -\frac{6i}{l}\theta_A - \frac{6i}{l}\theta_B + \frac{12i}{l^2}\Delta \ldots(2)$$

用力法求解单跨超静定梁

$$\delta_{11}X_1+\delta_{12}X_2+\Delta_{1C}=\theta_A$$

$$\delta_{21}X_1+\delta_{22}X_2+\Delta_{2C}=\theta_B$$

$$\delta_{11}=\frac{1}{EI}\frac{l}{2}\frac{2}{3}=\frac{l}{3EI}=\delta_{22}$$

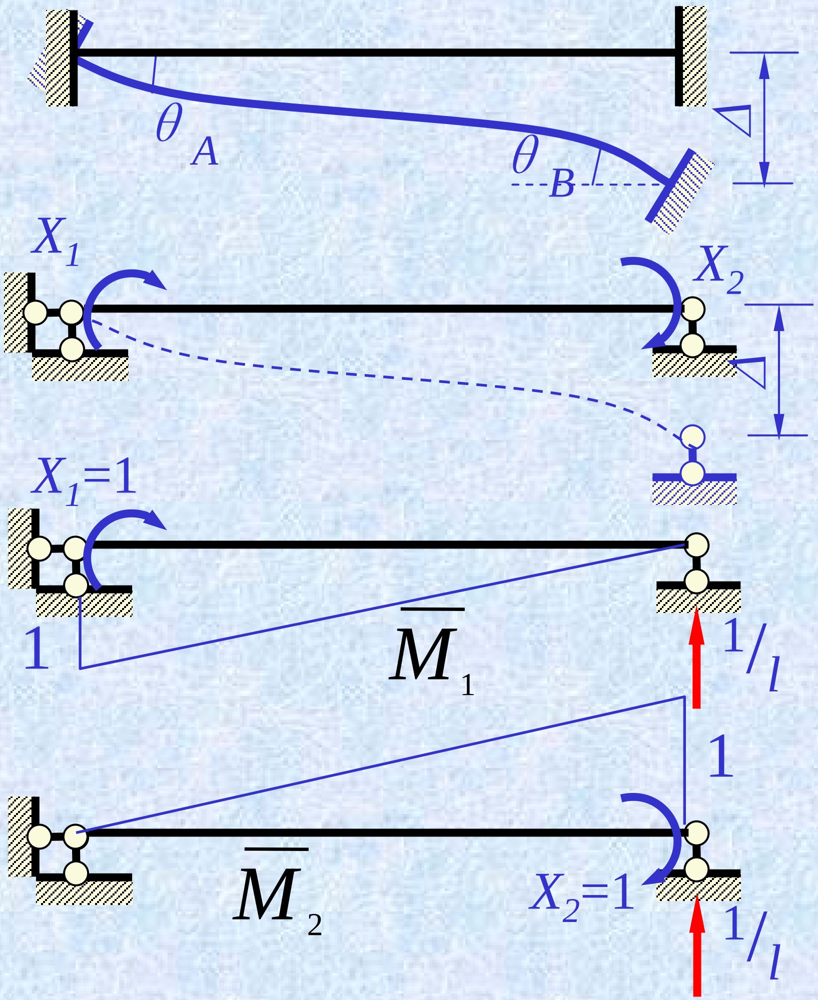
$$\delta_{12}=-\frac{1}{EI}\frac{l}{2}\frac{1}{3}=\frac{l}{6EI}=\delta_{21}$$

$$\Delta_{1C}=\frac{\Delta}{l}=\Delta_{2C}$$

$$\frac{l}{3EI}X_1-\frac{l}{6EI}X_2+\frac{\Delta}{l}=\theta_A$$

$$-\frac{l}{6EI}X_1+\frac{l}{3EI}X_2+\frac{\Delta}{l}=\theta_B$$

令 $i=\frac{EI}{l}$

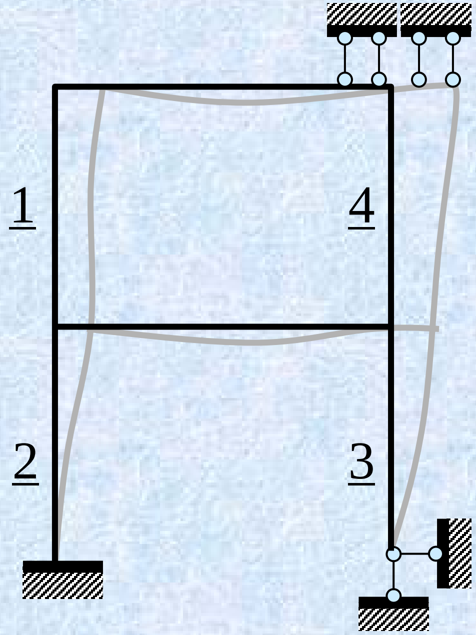


$$X_1=4i\theta_A+2i\theta_B-\frac{6i}{l}\Delta$$

$$X_2=2i\theta_A+4i\theta_B-\frac{6i}{l}\Delta$$

可以将上式写成矩阵形式

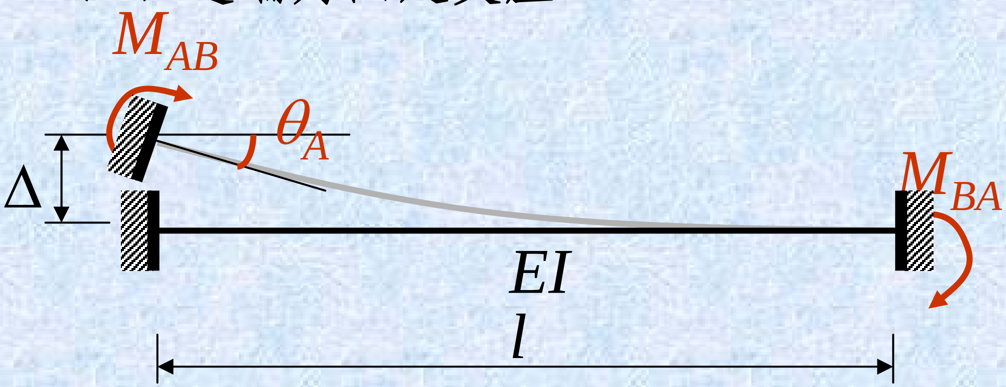
$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ Q_{AB} \end{bmatrix} = \begin{bmatrix} 4i & 2i & -\frac{6i}{l} \\ 2i & 4i & -\frac{6i}{l} \\ -\frac{6i}{l} & -\frac{6i}{l} & \frac{12i}{l^2} \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \Delta \end{bmatrix}$$



几种不同远端支座的刚度方程

$$\left.\begin{aligned} M_{AB} &= 4i\theta_A + 2i\theta_B - 6i\frac{\Delta}{l} \\ M_{BA} &= 2i\theta_A + 4i\theta_B - 6i\frac{\Delta}{l} \end{aligned}\right\} \cdots\cdots(1)$$

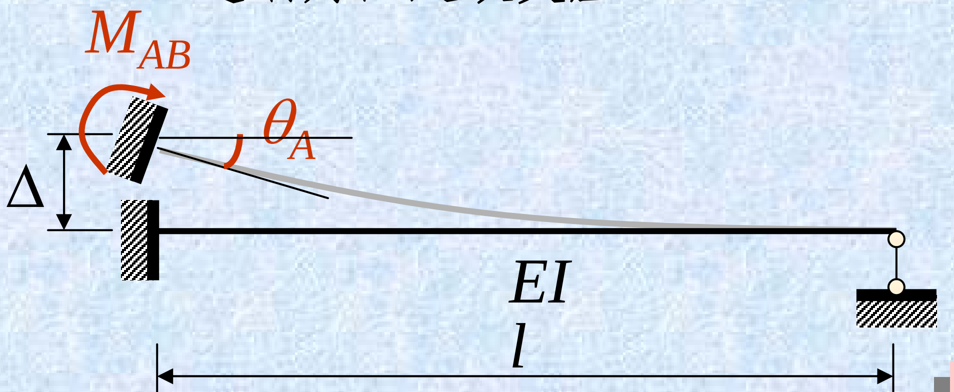
（1）远端为固定支座



因$\theta_B = 0$，代入(1)式可得

$$\left.\begin{aligned} M_{AB} &= 4i\theta_A - \frac{6i}{l}\Delta \\ M_{BA} &= 2i\theta_A - \frac{6i}{l}\Delta \end{aligned}\right\}$$

（2）远端为固定铰支座

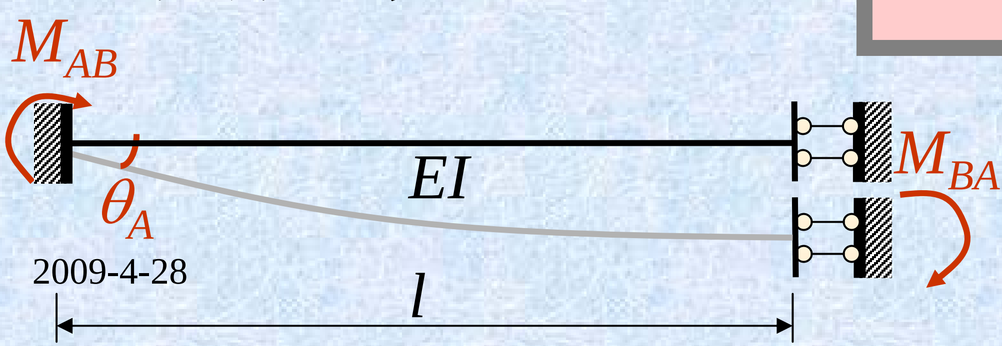


因$M_{BA} = 0$，代入(1)式可得

$$M_{AB} = 3i\theta_A - \frac{3i}{l}\Delta$$

$$Q_{AB} = Q_{BA} = -\frac{6i}{l}\theta_A - \frac{6i}{l}\theta_B + \frac{12i}{l^2}\Delta \ldots(2)$$

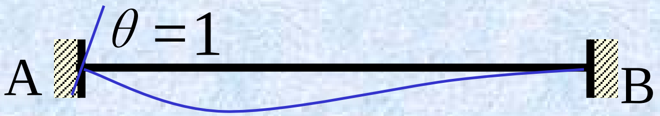
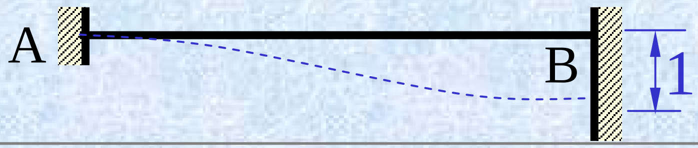
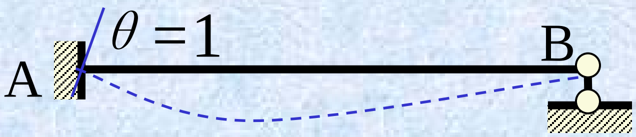
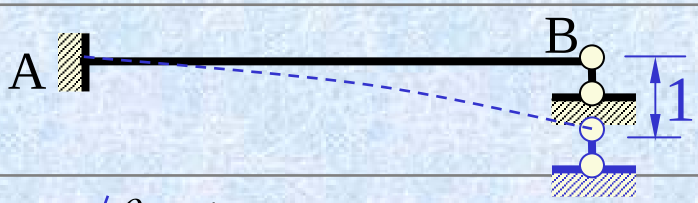
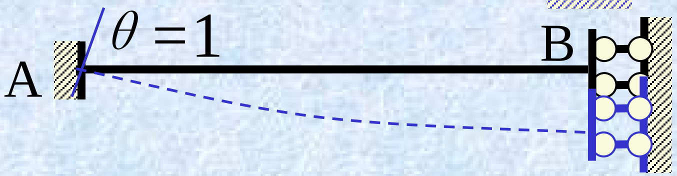
（3）远端为定向支座



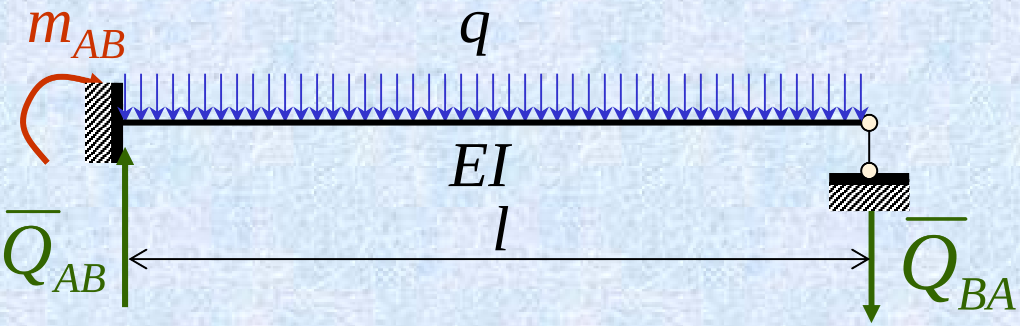
代入（2）式可得 $\frac{\Delta}{l} = \frac{1}{2}\theta_A$

$M_{AB} = i\theta_A$　　$M_{BA} = -i\theta_A$

由单位杆端位移引起的杆端力称为形常数。

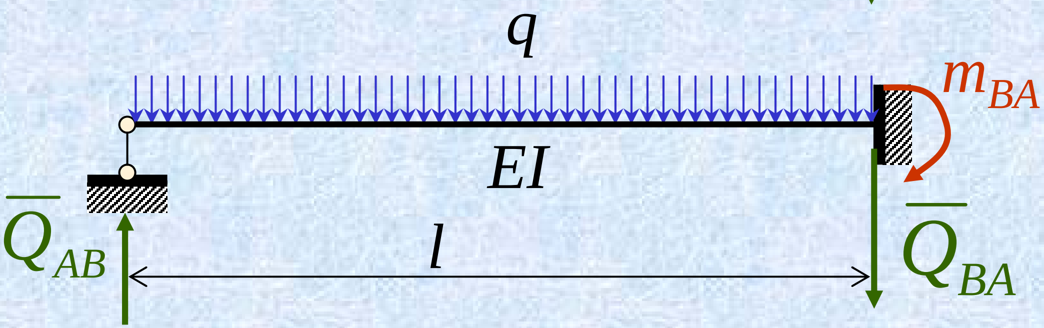
单跨超静定梁简图	M_{AB}	M_{BA}	$Q_{AB}=Q_{BA}$
	$4i$	$2i$	$-6i/l$
	$-6i/l$	$-6i/l$	$12i/l^2$
	$3i$	0	$-3i/l$
	$-3i/l$	0	$3i/l^2$
	i	$-i$	0

二、由荷载求固端反力



$$m_{AB}=-\frac{ql^2}{8}$$

$$\left.\begin{aligned}\overline{Q}_{AB}&=+\frac{5}{8}ql\\ \overline{Q}_{BA}&=-\frac{3}{8}ql\end{aligned}\right\}$$



$$m_{BA}=+\frac{ql^2}{8}$$

$$\left.\begin{aligned}\overline{Q}_{AB}&=+\frac{3}{8}ql\\ \overline{Q}_{BA}&=-\frac{5}{8}ql\end{aligned}\right\}$$

» 在已知荷载及杆端位移的共同作用下的杆端力一般公式（转角位移方程）：

$$M_{AB}=4i\theta_A+2i\theta_B-\frac{6i}{l}\Delta+m_{AB}$$

$$M_{BA}=2i\theta_A+4i\theta_B-\frac{6i}{l}\Delta+m_{BA}$$

$$Q_{AB}=-\frac{6i}{l}\theta_A-\frac{6i}{l}\theta_B+\frac{12i}{l^2}\Delta+\overline{Q}_{AB}$$

§11-3 位移法的基本体系

一、超静定结构计算的总原则:

欲求超静定结构先取一个基本体系,然后让基本体系在受力方面和变形方面与原结构完全一样。

力法的特点:

基本未知量——多余未知力;
基本体系——静定结构;
基本方程——位移条件
（变形协调条件）

位移法的特点:

基本未知量—— 独立结点位移
基本体系——一组单跨超静定梁
基本方程—— 平衡条件

二、基本未知量的选取

1、结点角位移数：

结构上可动刚结点数即为位移法计算的结点角位移数。

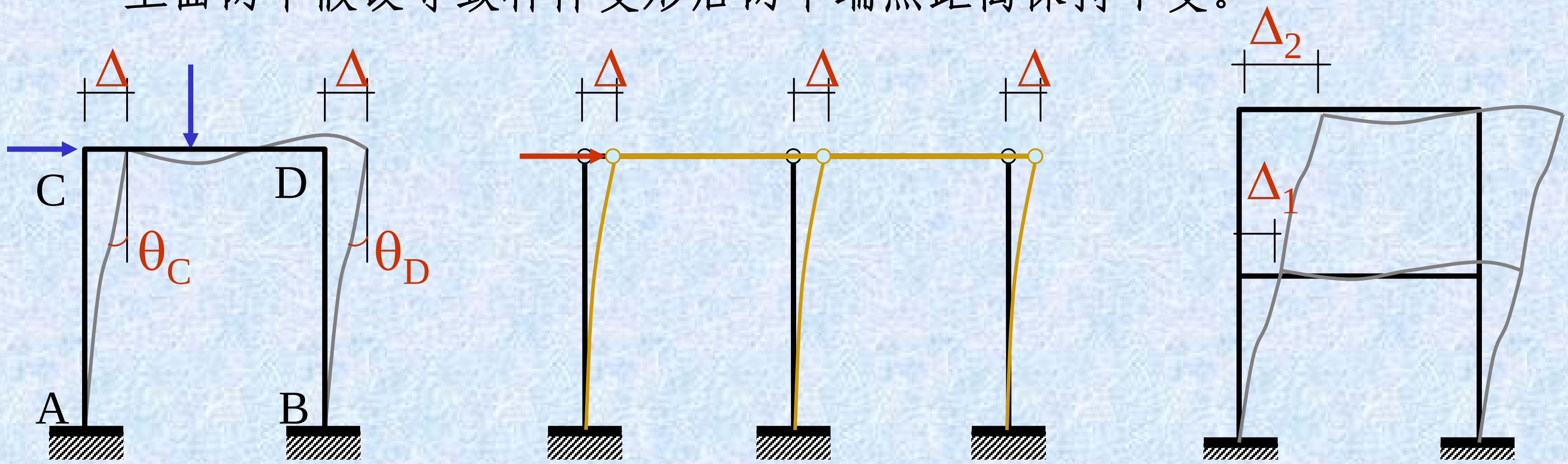
2、结构独立线位移：

每个结点有两个线位移，为了减少未知量，引入与实际相符的两个假设：

（1）忽略轴向力产生的轴向变形---变形后的曲杆与原直杆等长；

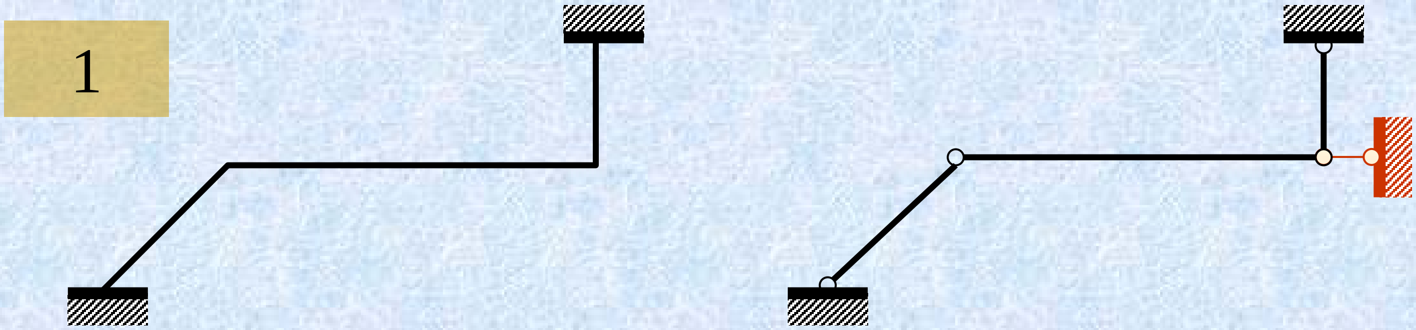
（2）变形后的曲杆长度与其弦等长。

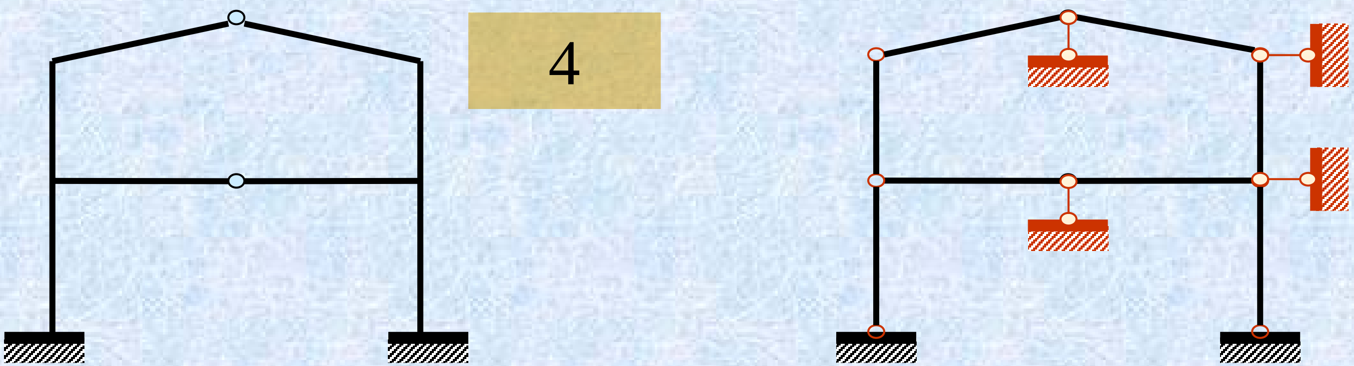
上面两个假设导致杆件变形后两个端点距离保持不变。

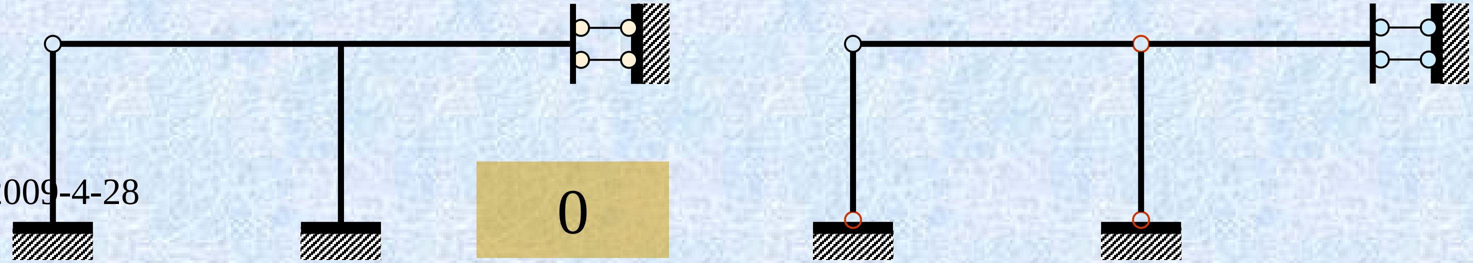


线位移数也可以用几何方法确定。

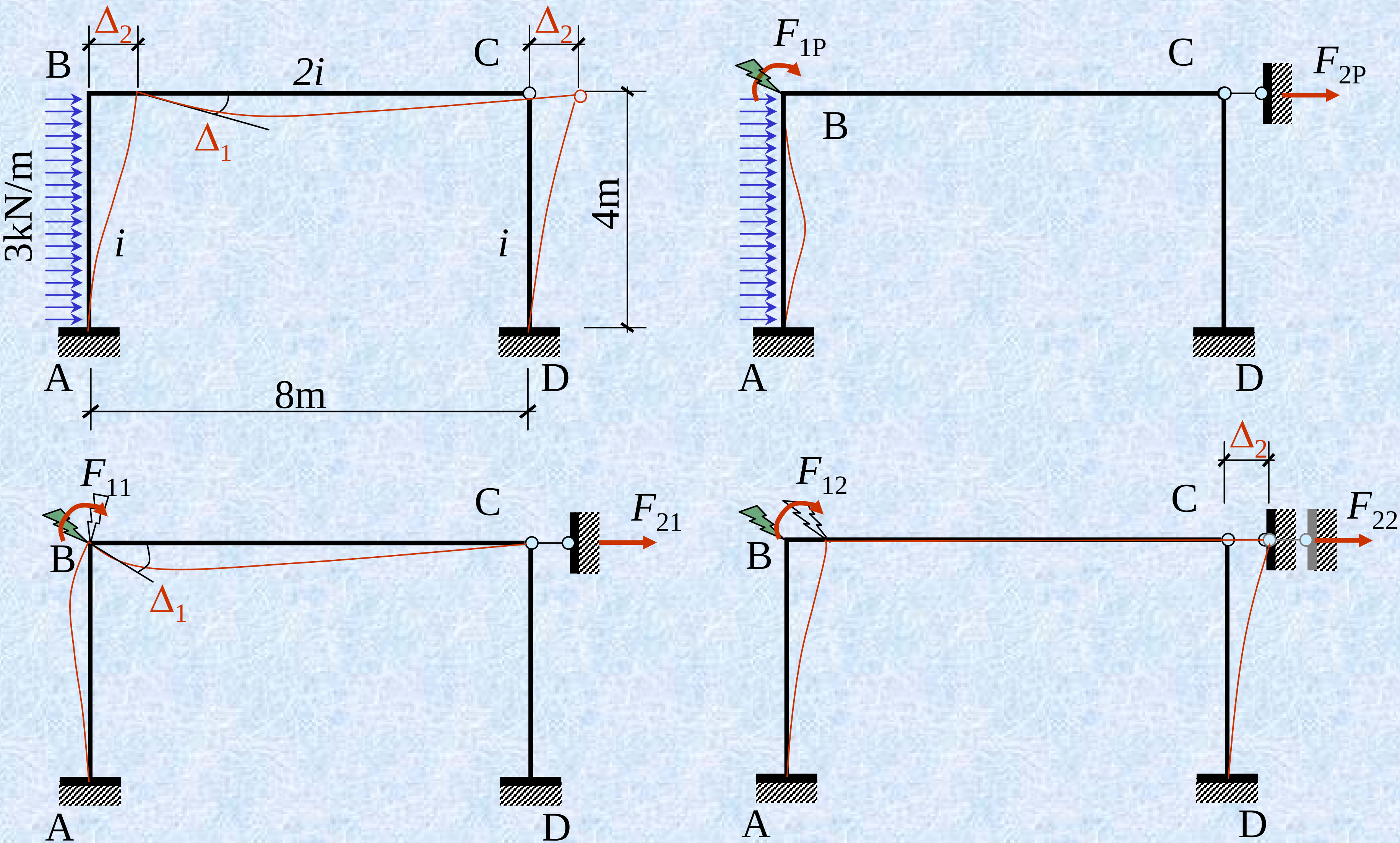
将结构中所有刚结点和固定支座，代之以铰结点和铰支座，分析新体系的几何构造性质，若为几何可变体系，则通过增加支座链杆使其变为无多余联系的几何不变体系，所需增加的链杆数，即为原结构位移法计算时的线位移数。







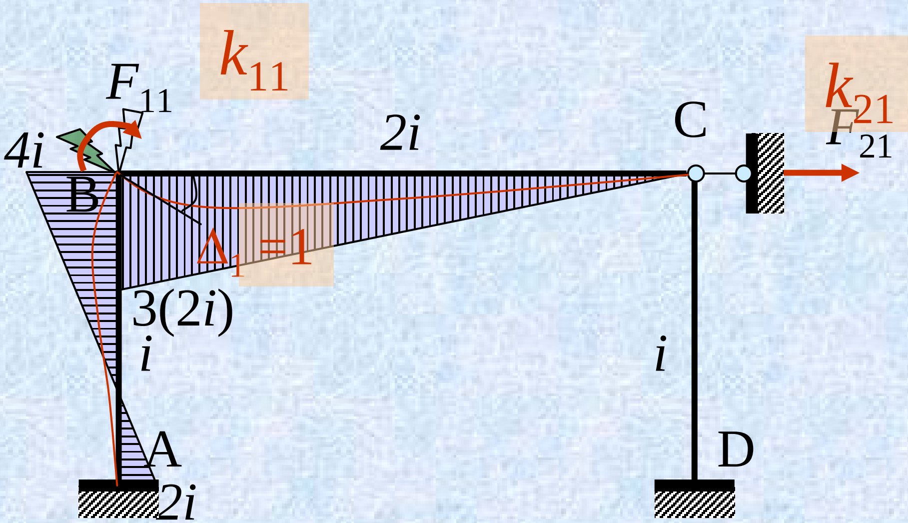
三、选择基本体系

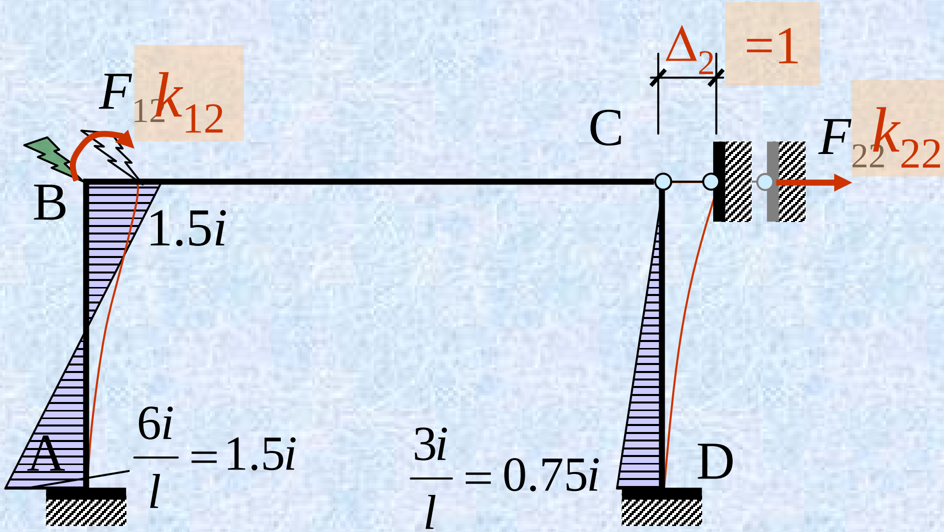


四、建立基本方程

$$F_{11}+F_{12}+F_{1P}=0 \ldots\ldots\ldots\ldots\ldots\ldots(1a)$$

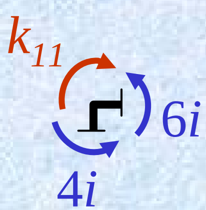
$$F_{21}+F_{22}+F_{2P}=0 \ldots\ldots\ldots\ldots\ldots\ldots(2a)$$

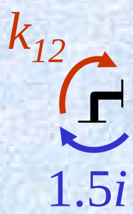




$$F_{11}+F_{12}+F_{1P}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1a)$$

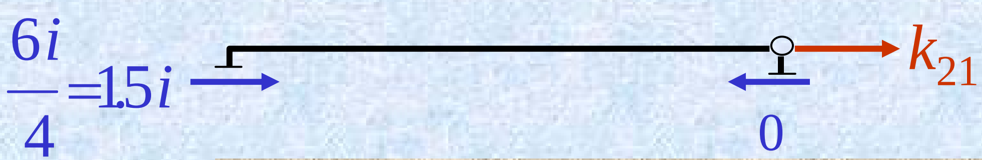
$$F_{21}+F_{22}+F_{2P}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2a)$$

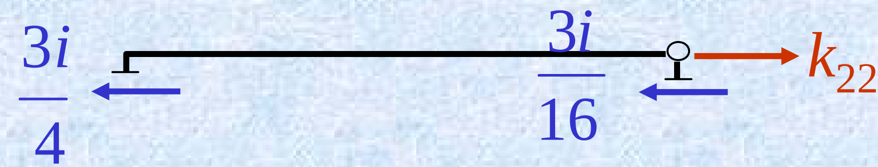




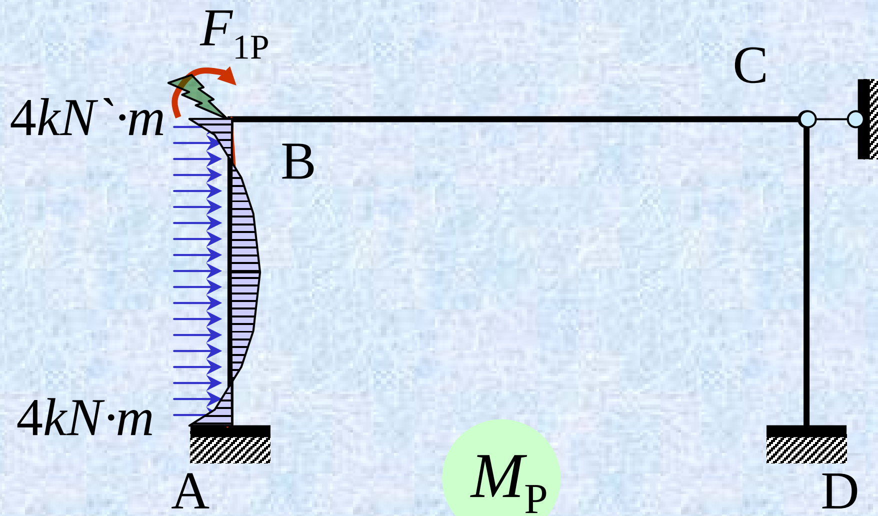
$$k_{11}\Delta_1 + k_{12}\Delta_2 + F_{1P} = 0\ldots\ldots\ldots(1)$$

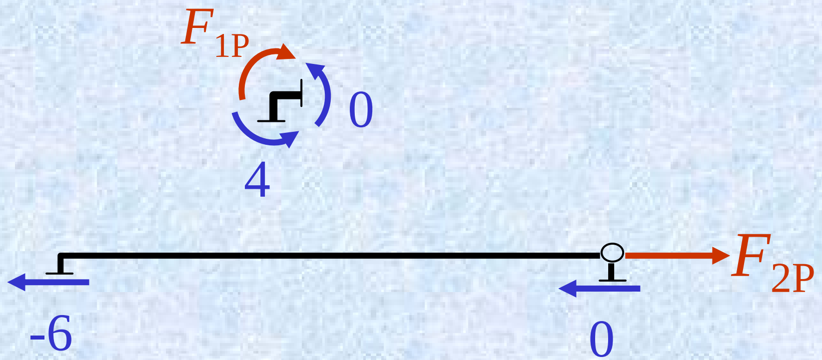
$$k_{21}\Delta_1 + k_{22}\Delta_2 + F_{2P} = 0\ldots\ldots\ldots(2)$$





$k_{11}=10i$ $\quad k_{21}= -1.5i$ $\quad k_{12}= -1.5i$ $\quad k_{22} = \dfrac{15}{16}i$





F_{1P}=4kN·m F_{2P}=-6kN

位移法方程:

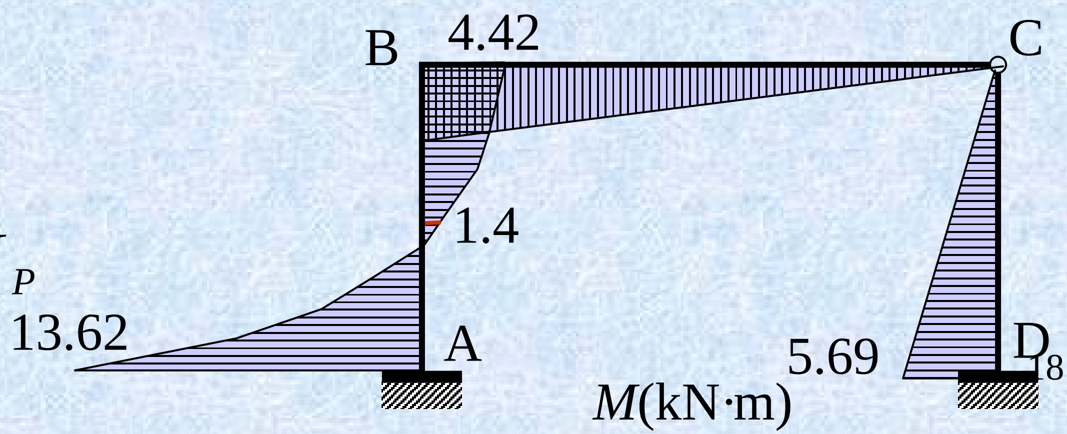
$$\left.\begin{aligned} 10\ i\Delta_1 - 1.5\ i\Delta_2 + 4 = 0 \\ -1.5i\Delta_1 + \frac{15}{16}i\Delta_2 - 6 = 0 \end{aligned}\right\}$$

$$\Delta_1 = 0.737\frac{1}{i} \qquad \Delta_2 = 7.580\frac{1}{i}$$

五、计算结点位移

六、绘制弯矩图

$$M = \overline{M}_1\Delta_1 + \overline{M}_2\Delta_2 + M_P$$

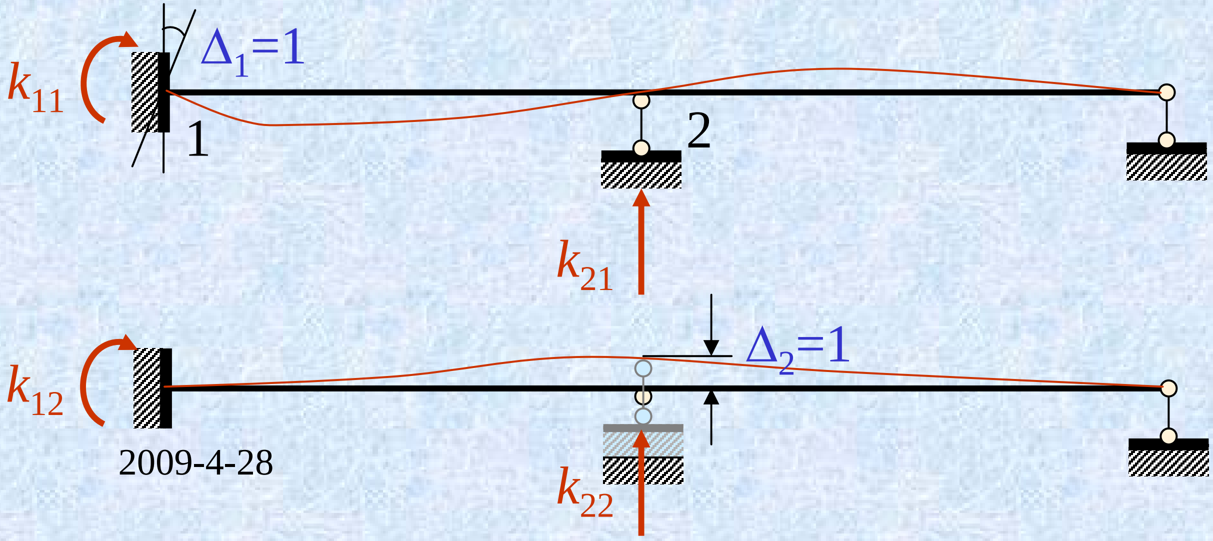


具有n个独立结点位移的超静定结构:

$$\left.\begin{aligned}
&k_{11}\Delta_1 + k_{12}\Delta_2 + \cdots\cdots + k_{1n}\Delta_n + F_{1P} = 0\\
&k_{21}\Delta_1 + k_{22}\Delta_2 + \cdots\cdots + k_{2n}\Delta_n + F_{2P} = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&k_{n1}\Delta_1 + k_{n2}\Delta_2 + \cdots\cdots + k_{nn}\Delta_n + F_{nP} = 0
\end{aligned}\right\}$$

$$\begin{bmatrix} k_{11} & k_{12} & \cdots\cdots & k_{1n} \\ k_{21} & k_{22} & \cdots\cdots & k_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ k_{n1} & k_{n2} & \cdots\cdots & k_{nn} \end{bmatrix}$$

$k_{ij}=k_{ji}$



$$k_{11}\times 0+k_{21}\times 1$$
$$= k_{12}\times 1+k_{22}\times 0$$

$k_{21}=k_{12}$