Differential Forms and the Noncommutative Residue for Manifolds with Boundary in the Non-product Case *

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Abstract In this paper, for an even dimensional compact manifold with boundary which has the non-product metric near the boundary, we use the noncommutative residue to define a conformal invariant pair. For a 4-dimensional manifold, we compute this conformal invariant pair under some conditions and point out the way of computations in the general.

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1 Introduction

Since the noncommutative residue was found in [Ad],[M],[Gu],[Wo], it was applied to many branches of mathematics. Especially, it was as the noncommutative counterpart of the integral in NCG by [C1]. The noncommutative residue also had been used to derive the gravitational action in the framework of NCG in [K], [KW]. In [C2], Connes used the noncommutative residue to find a conformal 4-dimensional Polyakov action analogy. In [U], Connes' result was generalized to the higher dimensional case.

The noncommutative residue on Boutet de Monvel algebra for manifolds with boundary was found in [FGLS]. In [S], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. In [Wa1], the author proved a Kastler-Kalau-Walze type theorem for manifolds with boundary and for the boundary flat case, he gave two kinds of operator theoretic explaination of the gravitational action on boundary. In [Wa2], the author generalized the results in [C2] and [U] to the case of manifolds with boundary which have a product metric near the

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boundary. A natural question is to define and compute a conformal invariant pair in the non-product metric case. In this paper, for an even dimensional compact manifold with boundary which has a non-product metric near the boundary, we define a conformal invariant pair. When $n = 4$, we compute this conformal invariant pair under some conditions and point out the way of computations in the general. As a corollary, when $n = 4$, for some special non-product metrics, we get the conformal invariant on the boundary vanishes which generalizes partially a result in [Wa2].

This paper is organized as follows: In Section 2, we define a conformal invariant pair associated to an even dimensional compact manifold with boundary which has a non-product metric near the boundary. In Section 3, for a 4-dimensional manifold, we compute this conformal invariant pair under some conditions. Some remarks on computations in the general case when $n = 4$ will be given in Section 4.

2 The Conformal Invariant Pair $(\Omega_n(f_1, f_2), \Omega_{n-1}(f_1, f_2))$

Let M be an even dimensional compact oriented Riemaniann manifold with boundary ∂M and $U \subset M$ be the collar neighborhood of ∂M which is diffeomorphic to $\partial M \times [0, 1)$. Write $\dim M = n$. Let g^M be the metric on M which has the following form on U

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \tag{2.1}$$

where $g^{\partial M}$ is the metric on ∂M; $h(x_n) \in C^\infty([0, 1)) = \{g|_{[0,1)} | g \in C^\infty((-\varepsilon, 1))\}$ for some $\varepsilon > 0$ and satisfies $h(x_n) > 0,\ h(0) = 1$ where x_n denotes the normal directional coordinate.

In this section, we will construct a conformal invariant pair $(\Omega_n(f_1, f_2), \Omega_{n-1}(f_1, f_2))$ associated to M. The fundamental setup is the same as Section 2 and Section 3 in [Wa2]. Recall that in Section 4 of [Wa2], we consider the product metric case, i.e. $h(x_n) \equiv 1$. We can use a canonical way to construct a metric $\widetilde{g}$ on the double manifold $\widehat{M} = M \cup_{\partial M} M$ through taking $\widetilde{g} = g$ on both copies of M, then $\widetilde{g}$ is well defined by $h = 1$. But for the general h, this is not correct. So we need to use another way to construct a conformal invariant pair associated to M.

By the definition of $C^\infty([0, 1))$ and $h > 0$, there exists $\widetilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\widetilde{h}|_{[0,1)} = h$ and $\widetilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Using partition of unity Theorem, then there exists a metric $\widehat{g}$ on $\widehat{M}$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$g^M = \frac{1}{\widetilde{h}(x_n)} g^{\partial M} + dx_n^2, \tag{2.2}$$

such that $\widehat{g}|_M = g$. Nextly we fix a metric $\widehat{g}$ on the $\widehat{M}$ such that $\widehat{g}|_M = g$. Denote by $[(M, g)]$ a conformal manifold. The way of constructing a conformal invariant pair associated to $[(M, g)]$ is as follows. As in [C2] or [U], we consider the following operator on the manifold $(\widehat{M}, \widehat{g})$,

$$F_{\widehat{g}} := \frac{d\delta - \delta d}{d\delta + \delta d} : \wedge^{\frac{n}{2}}(T^*\widehat{M}) \to \wedge^{\frac{n}{2}}(T^*\widehat{M}), \tag{2.3}$$

then $F_{\widehat{g}}$ does not depend on the choice of the metric in the conformal class $[(\widehat{M},\widehat{g})]$. Now similar to (3.5) and (3.6) in [Wa2], for $f_0, f_1, f_2 \in C^\infty(M)$ and f_0 not depending on x_n near the boundary, we define the form pair $(\Omega_n(f_1,f_2)(\widehat{g}), \Omega_{n-1}(f_1,f_2)(\widehat{g}))$ through the following equality:

$$\widetilde{\mathrm{Wres}}(\pi^+ f_0[\pi^+ F_{\widehat{g}}, \pi^+ f_1][\pi^+ F_{\widehat{g}}, \pi^+ f_2])$$
$$= \int_M f_0 \Omega_n(f_1,f_2)(\widehat{g}) + \int_{\partial M} f_0|_{\partial M} \Omega_{n-1}(f_1,f_2)(\widehat{g}). \tag{2.4}$$

By the definition of $\pi^+ F_{\widehat{g}}$ in the Boutet de Monvel algebra, the left term of (2.4) is well defined. We hope to generalize the results in [C2] and [U], so as in [U], we take $\Omega_n(f_1,f_2)(\widehat{g}) =$

$$\int_{|\xi|=1} \mathrm{tr}\left[\sum \frac{1}{\alpha'!\alpha''!\beta!\delta!} D_x^\beta \bar{f}_1 D_x^{\alpha''+\delta} \bar{f}_2 \partial_\xi^{\alpha'+\alpha''+\beta} \sigma_{-j}^{F_{\widehat{g}}} \partial_\xi^\delta D_x^{\alpha'} \sigma_{-k}^{F_{\widehat{g}}}\right] \sigma(\xi) d^n x\, |_M\ , \tag{2.5}$$

where $\sigma_{-j}^{F_{\widehat{g}}}$ denotes the order $-j$ symbol of $F_{\widehat{g}}$; $\overline{f_1}$, $\overline{f_2}$ are the extensions to $\widehat{M}$ of f_1, f_2, $D_x^\beta = (-i)^{|\beta|}\partial_x^\beta$, and the sum is taken over $|\alpha'| + |\alpha''| + |\beta| + |\delta| + j + k = n; |\beta| \geq 1, |\delta| \geq 1; \alpha', \alpha'', \beta, \delta \in \mathbf{Z}_+^n; j, k \in \mathbf{Z}_+$. Then $\Omega_n(f_1,f_2)(\widehat{g})$ does not depend on the extensions of f_1, f_2. By Theorem 3.1 and (3.19) in [Wa2], then $\Omega_{n-1}(f_1,f_2)(\widehat{g})$ is uniquely determined by (2.4), (2.5) as follows:

$$\Omega_{n-1}(f_1,f_2)(\widehat{g}) = \sum_{j,k=0}^{\infty} \sum \sum_{|\beta|=1}^{-r} \sum_{|\delta|=1}^{-l} \frac{(-i)^{j+k+1+|\alpha|+|\beta|+|\delta|}}{\alpha!\beta!\delta!(j+k+1)!}$$
$$\times \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}\left\{ \partial_{x_n}^j \left[\partial_x^\beta f_1 \partial_{\xi'}^\alpha \partial_{\xi_n}^k \pi_{\xi_n}^+ \partial_\xi^\beta \sigma_{r+|\beta|}^{F_{\widehat{g}}}\right] |_{x_n=0}\right.$$
$$\left. \times \partial_{x'}^\alpha \partial_{x_n}^k \left[\partial_x^\delta f_2 \partial_{\xi_n}^{j+1} \partial_\xi^\delta \sigma_{l+|\delta|}^{F_{\widehat{g}}}\right] |_{x_n=0} \right\} d\xi_n \sigma(\xi') d^{n-1}x', \tag{2.6}$$

where the sum is taken over $r - k - |\alpha| + l - j - 1 = -n, \quad r, l \leq -1, \quad |\alpha| \geq 0$. Then we have

Theorem 2.1 *The form pair $(\Omega_n(f_1,f_2)(\widehat{g}), \Omega_{n-1}(f_1,f_2)(\widehat{g}))$ only depends on g and does not depend on the extension $\widehat{g}$. It is a uniquely determined conformal invariant pair on $[(M,g)]$ by (2.4), (2.5), and is symmetric in f_1 and f_2.*

Proof. By (2.5), (2.6), in order to prove that the form pair $(\Omega_n(f_1,f_2)(\widehat{g}), \Omega_{n-1}(f_1,f_2)(\widehat{g}))$ only depends on g and does not depend on the extension $\widehat{g}$, we only need to prove that $D_x^\alpha(\sigma_{-j}^{F_{\widehat{g}}})|_M$ and $D_x^\alpha(\sigma_{-j}^{F_{\widehat{g}}})|_{x_n=0}$ do not depend on the extension $\widehat{g}$. By Lemma A.3 in [U], this is equivalent to proving that $D_x^\alpha(\widehat{g}_{i,j})|_M$ and $D_x^\alpha(\widehat{g}_{i,j})|_{x_n=0}$ do not depend on the extension $\widehat{g}$, where $[\widehat{g}_{i,j}]$ is the metric matrix of $\widehat{g}$. The latter is trivial, so we prove the first assertion. This fact says that $(\Omega_n(f_1,f_2)(\widehat{g}), \Omega_{n-1}(f_1,f_2)(\widehat{g}))$ is a form pair with coefficients of derivatives of $g_{i,j}$, so we can write $(\Omega_n(f_1,f_2)(g), \Omega_{n-1}(f_1,f_2)(g))$ instead of $(\Omega_n(f_1,f_2)(\widehat{g}), \Omega_{n-1}(f_1,f_2)(\widehat{g}))$.

By (2.5),(2.6), in order to prove $(\Omega_n(f_1,f_2)(g),\Omega_{n-1}(f_1,f_2)(g))$ is a conformal invariant of $[(M,g)]$, we only need prove $\int_{|\xi|=1}p_{-n}(x,\xi)\sigma(\xi)$; $\int_{|\xi'|=1}p'_{-n+1}(x',\xi')\sigma(\xi')$ where $p_{-n}(x,\xi)$ $(p'_{-n+1}(x',\xi'))$ is a homogeneous function of degree $-n$ $(-n+1)$ about ξ (ξ') and $D^\alpha_x(\sigma^{F_{\widehat{g}}}_{-j})|_M$; $D^\alpha_x(\sigma^{F_{\widehat{g}}}_{-j})|_{x_n=0}$ do not depend on the choice of the representative of $[(M,g)]$. As the discussions in [AM], $\int_{|\xi|=1}p_{-n}(x,\xi)\sigma(\xi)$; $\int_{|\xi'|=1}p'_{-n+1}(x',\xi')\sigma(\xi')$ do not depend on the choice of metric. For any representative e^fg of $[(M,g)]$ where $f\in C^\infty(M)$, since $(\Omega_n(f_1,f_2)(g),\Omega_{n-1}(f_1,f_2)(g))$ does not depend on the extension $\widehat{g}$, we can choose the extension $e^{\widehat{f}}\widehat{g}$ of e^fg to compute $(\Omega_n(f_1,f_2)(e^fg)$, $\Omega_{n-1}(f_1,f_2)(e^fg))$ where $\widehat{f}\in C^\infty(\widehat{M})$ is an extension of f. By $F_{\widehat{g}}=F_{\widehat{e^fg}}$, so symbols $\sigma(F_{\widehat{g}})=\sigma(F_{\widehat{e^fg}})$. Then by (2.5) and (2.6),

$$(\Omega_n(f_1,f_2)(g),\Omega_{n-1}(f_1,f_2)(g))=(\Omega_n(f_1,f_2)(e^fg),\Omega_{n-1}(f_1,f_2)(e^fg)).$$

The other properties of $(\Omega_n(f_1,f_2)(g),\Omega_{n-1}(f_1,f_2)(g))$ come from Theorem 3.1 and Proposition 3.3 in [Wa2]. □

3 The Computation of $(\Omega_4(f_1,f_2),\Omega_3(f_1,f_2))$

In this section, we want to compute $(\Omega_4(f_1,f_2),\Omega_3(f_1,f_2))$ defined in Section 2 when $n=4$. We hope to compare the change of $(\Omega_4(f_1,f_2),\Omega_3(f_1,f_2))$ under the product metric and the nonproduct metric. So for simplicity, we firstly assume that $(\star)$ $f_1,\ f_2$ *are independent of* x_n *near the boundary.* For the general case, we will point out the way of computations in Section 4.

$\Omega_4(f_1,f_2)$ is computed by Theorem 4.5 in [Wa2]. By (2.6) and the assumption $(\star)$, then

$$\Omega_3(f_1,f_2)=\sum_{j,k=0}^{\infty}\sum\sum_{|\beta'|=1}^{-r}\sum_{|\delta'|=1}^{-l}\frac{(-i)^{j+k+1+|\alpha|+|\beta'|+|\delta'|}}{\alpha!\beta'!\delta'!(j+k+1)!}$$
$$\times\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}_{\wedge^2T^*M}\left\{\left[\partial^{\beta'}_{x'}f_1\times\partial^j_{x_n}\partial^{\alpha+\beta'}_{\xi'}\partial^k_{\xi_n}\pi^+_{\xi_n}\sigma^{F_{\widehat{g}}}_{r+|\beta'|}\right]|_{x_n=0}\right.$$
$$\left.\times\partial^\alpha_{x'}\left[\partial^{\delta'}_{x'}f_2\partial^k_{x_n}\partial^{j+1}_{\xi_n}\partial^{\delta'}_{\xi'}\sigma^{F_{\widehat{g}}}_{l+|\delta'|}\right]|_{x_n=0}\right\}d\xi_n\sigma(\xi')d^{n-1}x',\tag{3.1}$$

where the sum is taken over $-(r+l)+|\alpha|+k+j=3,\ \ r,l\leq-1,\ \ \alpha,\beta',\delta'\in\mathbf{Z}^3_+$. Since $\Omega_3(f_1,f_2)$ is a global form on ∂M, so for any fixed point $x_0\in\partial M$, we can choose the normal coordinates V of x_0 in ∂M (not in M) and compute $\Omega_3(f_1,f_2)(x_0)$ in the coordinates $x=(x',x_n)=(x_1,\cdots,x_{n-1},x_n)$ and domain $\widetilde{V}=V\times[0,1)\subset M$ and the metric $\frac{1}{h(x_n)}g^{\partial M}+dx_n^2$. The dual metric of g^M on $\widetilde{V}$ is $h(x_n)g^{\partial M}+dx_n^2$. Write $g^M_{ij}=g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$; $g^{ij}_M=g^M(dx_i,dx_j)$, then

$$[g^M_{i,j}]=\begin{bmatrix}\frac{1}{h(x_n)}[g^{\partial M}_{i,j}] & 0\\ 0 & 1\end{bmatrix};\quad [g^{i,j}_M]=\begin{bmatrix}h(x_n)[g^{i,j}_{\partial M}] & 0\\ 0 & 1\end{bmatrix},$$

and

$$\partial_{x_s} g_{ij}^{\partial M}(x_0)=0, 1\leq i,j\leq n-1;\quad g_{ij}^{M}(x_0)=\delta_{ij}. \tag{3.2}$$

We'll compute $\mathrm{tr}_{\wedge^2(T^*M)}$ in the frame $\{dx_{i_1}\wedge dx_{i_2}|\ 1\leq i_1<i_2\leq 4\}$, which is independent of the choice of frames. Let $\epsilon(\xi),\ \iota(\xi)$ be the exterior and interior multiplications respectively where $\xi=\sum_{i=1}^{n}\xi_i dx_i$ denotes a cotangent vector. Recall Lemma 2.2 in [Wa1]

$$\partial_{x_j}(|\xi|^2_{g^M})(x_0)=0,\ \text{if } j<n;\ \partial_{x_n}(|\xi|^2_{g^M})(x_0)=h'(0)|\xi'|^2_{g^{\partial M}}. \tag{3.3}$$

By (3.2) and $h(0)=1$, then under the frame $\{dx_{i_1}\wedge dx_{i_2}|\ 1\leq i_1<i_2\leq 4\}$, $\partial_{x_i}\varepsilon(dx_j)=0$ and

$$\partial_{x_l}\iota(dx_j)(x_0)=0,\ \text{if } l<n;\ \partial_{x_n}\iota(dx_j)(x_0)=h'(0)\iota(dx_j)(x_0). \tag{3.4}$$

So if $i<n$, then

$$\partial_{x_i}\varepsilon(\xi)(x_0)=\partial_{x_i}\iota(\xi)(x_0)=0;\ \partial_{x_n}\iota(\xi)(x_0)=h'(0)\iota(\xi')(x_0). \tag{3.5}$$

Theorem 3.1 *Under the above conditions,*

$$\Omega_3(f_1,f_2)(x_0)=h'(0)\sum_{1\leq i,j\leq 3}a_{i,j}\partial_{x_i}f_1\partial_{x_j}f_2dx_1\wedge dx_2\wedge dx_3, \tag{3.6}$$

where $a_{i,j}$ is a constant.

Corollary 3.2 *Under the assumption $(\star)$, if $h'(0)=0$ (for example $h=1-x_n^2$), then $\Omega_3(f_1,f_2)=0$. Especially, if g^M has the product metric near the boundary, then $\Omega_3(f_1,f_2)=0$ and*

$$\widetilde{\mathrm{Wres}}(\pi^+f_0[\pi^+F_{\widehat{g}},\pi^+f_1][\pi^+F_{\widehat{g}},\pi^+f_2])=\int_M f_0\Omega_4(f_1,f_2). \tag{3.7}$$

Now we prove Theorem 3.1. Since the sum is taken over $-(r+l)+|\alpha|+k+j=3,\ \ r,l\leq -1$, so $\Omega_3(f_1,f_2)$ is the sum of the following five cases.

case a) I) $r=-1,\ l=-1\ k=j=0,\ |\alpha|=1$

For convenience, we use F instead of $F_{\widehat{g}}$ in the following. Let σ_L^F denote the leading symbol of F. By (3.1), we get

$$\text{case a) I)}=\sum_{|\alpha|=1}\sum_{|\beta'|=1}\sum_{|\delta'|=1}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}_{\wedge^2T^*M}\Big\{\partial_{x'}^{\beta'}f_1\partial_{\xi'}^{\alpha+\beta'}\pi^+_{\xi_n}\sigma_L^F$$

$$\times\Big[\partial_{x'}^{\alpha+\delta'}f_2\partial_{\xi_n}\partial_{\xi'}^{\delta'}\sigma_L^F+\partial_{x'}^{\delta'}f_2\partial_{x'}^{\alpha}\partial_{\xi_n}\partial_{\xi'}^{\delta'}\sigma_L^F\Big]\Big\}(x_0)d\xi_n\sigma(\xi')d^{n-1}x'. \tag{3.8}$$

It is necessary to compute

$$\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi'}^{\alpha+\beta'}\pi^+_{\xi_n}\sigma_L^F\times\partial_{\xi_n}\partial_{\xi'}^{\delta'}\sigma_L^F](x_0)$$

and

$$\mathrm{trace}_{\wedge^2 T^*M}[\partial_{\xi'}^{\alpha+\beta'}\pi^+_{\xi_n}\sigma_L^F \times \partial_{\xi_n}\partial_{\xi'}^{\delta'}\partial_{x'}^{\alpha}\sigma_L^F](x_0).$$

Using the computations in [Wa2,p.17], for $l,i,j<n$, then

$$\partial_{\xi_l}\partial_{\xi_i}\partial_{\eta_j}\left\{\mathrm{trace}\left[\pi^+_{\xi_n}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]\right\}|_{\xi'=\eta'}=\sum\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2k+1}}f(\xi_n),$$

where $f(\xi_n)$ is a smooth function about ξ_n and $1\leq i_1,\cdots,i_{2k+1}<n$. Integration over $|\xi'|=1$ is zero. By (3.3) and (3.5), then

$$\partial_{x_i}\sigma_L^F(x_0)=\partial_{x_i}\left[\frac{\varepsilon(\xi)\iota(\xi)-\iota(\xi)\varepsilon(\xi)}{|\xi|^2}\right](x_0)=0,$$

so case a) I) yields zero.

case a) II) $r=-1,\ l=-1\ k=|\alpha|=0,\ j=1$

By (3.1), we get

$$\text{case a) II)}=\frac{1}{2}\sum_{|\beta'|=1}\sum_{|\delta'|=1}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{x'}^{\beta'}f_1\partial_{x'}^{\delta'}f_2$$

$$\times\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi'}^{\beta'}\pi^+_{\xi_n}\partial_{x_n}\sigma_L^F\times\partial_{\xi'}^{\delta'}\partial_{\xi_n}^2\sigma_L^F](x_0)d\xi_n\sigma(\xi')d^{n-1}x'. \tag{3.9}$$

case a) III) $r=-1,\ l=-1\ j=|\alpha|=0,\ k=1$

By (3.1), we get

$$\text{case a) III)}=\frac{1}{2}\sum_{|\beta'|=1}\sum_{|\delta'|=1}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{x'}^{\beta'}f_1\partial_{x'}^{\delta'}f_2$$

$$\times\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi'}^{\beta'}\partial_{\xi_n}\pi^+_{\xi_n}\sigma_L^F\times\partial_{\xi'}^{\delta'}\partial_{\xi_n}\partial_{x_n}\sigma_L^F](x_0)d\xi_n\sigma(\xi')d^{n-1}x'. \tag{3.10}$$

Write

$$p(\xi)=\varepsilon(\xi)\iota(\xi)-\iota(\xi)\varepsilon(\xi).$$

By (3.3),(3.4),(3.5), then

$$\partial_{x_n}p(\xi)(x_0)=h'(0)[\varepsilon(\xi)\iota(\xi')-\iota(\xi')\varepsilon(\xi)](x_0);$$

$$\partial_{x_n}\sigma_L^F(x_0)=\frac{h'(0)[\varepsilon(\xi)\iota(\xi')-\iota(\xi')\varepsilon(\xi)](x_0)}{|\xi|^2}-\frac{h'(0)|\xi'|^2p(\xi)}{|\xi|^4}.$$

So case a) II+III) has the form in Theorem 3.1.

case b) $r=-2,\ l=-1,\ k=j=|\alpha|=0$

By (3.1), we get

$$\text{case b)} = \sum_{|\beta'|=1}^{2}\sum_{|\delta'|=1}\frac{(-i)^{2+|\beta'|}}{\beta'!}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{x'}^{\beta'}f_1\partial_{x'}^{\delta'}f_2$$

$$\times\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi'}^{\beta'}\pi_{\xi_n}^{+}\sigma_{-2+|\beta'|}^{F}\times\partial_{\xi'}^{\delta'}\partial_{\xi_n}\sigma_L^F](x_0)d\xi_n\sigma(\xi')d^{n-1}x'. \tag{3.11}$$

When $|\beta'|=2$, the term

$$\partial_{\xi_l}\partial_{\xi_i}\partial_{\eta_j}\left\{\mathrm{trace}\left[\pi_{\xi_n}^{+}\sigma_L(F)(\xi',\xi_n)\times\partial_{\xi_n}\sigma_L(F)(\eta',\xi_n)\right]\right\}|_{\xi'=\eta'}$$

will appear, as the disscusions in line 4 on p.6, it is zero after the integration over $|\xi'|=1$. So $|\beta'|=1$. In the following, we prove that $\sigma_{-1}(F)(x_0)$ has the coefficient $h'(0)$. Write $F=\frac{A}{\triangle}$, where $A=d\delta-\delta d$, $\triangle=d\delta+\delta d$, then by the composition formula of the symbol, we have

$$\sigma(F)=\sum_{|\alpha|\geq 0}\sum_{0\leq i\leq 2}\sum_{j\geq 2}\frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_i(A))D_x^\alpha(\sigma_{-j}(\triangle^{-1}));$$

$$\sigma_{-1}(F)=\sigma_1(A)\sigma_{-2}(\triangle^{-1})+\sigma_2(A)\sigma_{-3}(\triangle^{-1})+\sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_2(A))D_x^\alpha(\sigma_{-2}(\triangle^{-1})). \tag{3.12}$$

By (3.3), then

$$\sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_2(A))D_x^\alpha(\sigma_{-2}(\triangle^{-1}))(x_0)=\frac{ih'(0)|\xi'|^2\partial_{\xi_n}p(\xi)}{|\xi|^4}. \tag{3.13}$$

Similar to (3.12), then

$$\sigma_1(d\delta)=\sigma_1(d)\sigma_0(\delta)+\sigma_0(d)\sigma_1(\delta)-\sqrt{-1}\sum_i\partial_{\xi_i}(\sigma_1(d))\partial_{x_i}(\sigma_1(\delta));$$

$$\sigma_1(\delta d)=\sigma_1(\delta)\sigma_0(d)+\sigma_0(\delta)\sigma_1(d)-\sqrt{-1}\sum_i\partial_{\xi_i}(\sigma_1(\delta))\partial_{x_i}(\sigma_1(d)). \tag{3.14}$$

Let $\{e_1,\cdots,e_{n-1}\}$ be the orthonormal frame field in V about $g^{\partial M}$ which is parallel along geodesics and $e_i(x_0)=\frac{\partial}{\partial x_i}(x_0)$, then $\{\widetilde{e_1}=\sqrt{h(x_n)}e_1,\cdots,\widetilde{e_{n-1}}=\sqrt{h(x_n)}e_{n-1},\widetilde{e_n}=dx_n\}$ is the orthonormal frame field in $\widetilde{V}$ about g^M. By Lemma 2.3 and Section 3 in [Wa1], we have

$$\sigma_1(d)=\sqrt{-1}\varepsilon(\xi),\ \sigma_0(d)(x_0)=\frac{1}{4}h'(0)\sum_{i=1}^{n-1}\varepsilon(e_i^*)[\bar{c}(e_n)\bar{c}(e_i)-c(e_n)c(e_i)]; \tag{3.15}$$

$$\sigma_1(\delta)=-\sqrt{-1}\iota(\xi),\ \sigma_0(\delta)(x_0)=-\frac{1}{4}h'(0)\sum_{i=1}^{n-1}\iota(e_i^*)[\bar{c}(e_n)\bar{c}(e_i)-c(e_n)c(e_i)], \tag{3.16}$$

where

$$c(e_j)=\varepsilon(e_j^*)-\iota(e_j^*),\ \bar{c}(e_j)=\varepsilon(e_j^*)+\iota(e_j^*).$$

By (3.5), then

$$\sigma_1(d\delta)(x_0)=\sqrt{-1}\varepsilon(\xi)\sigma_0(\delta)(x_0)-\sqrt{-1}\sigma_0(d)(x_0)\iota(\xi)-\sqrt{-1}h'(0)\varepsilon(dx_n)\iota(\xi')(x_0); \quad (3.17)$$

$$\sigma_1(\delta d)(x_0)=-\sqrt{-1}\iota(\xi)\sigma_0(d)(x_0)+\sqrt{-1}\sigma_0(\delta)(x_0)\varepsilon(\xi). \quad (3.18)$$

By Lemma A.1 in [U] and (3.3), then

$$\sigma_{-3}(\triangle^{-1})(x_0)=-\frac{1}{|\xi|^2}[\sigma_1(\triangle)\frac{1}{|\xi|^2}-\sqrt{-1}\sum_i\partial_{\xi_i}(|\xi|^2)\partial_{x_i}(\frac{1}{|\xi|^2})](x_0)$$

$$=-\frac{\sigma_1(\triangle)(x_0)}{|\xi|^4}-\frac{2\sqrt{-1}h'(0)|\xi'|^2\xi_n}{|\xi|^6}. \quad (3.19)$$

By (3.12), (3.13), (3.15)-(3.19) and the definitions of $A,\ \triangle$, we get $\sigma_{-1}(F)(x_0)=h'(0)f(\xi)$. So case b) has the form in Theorem 3.1.

case c) $r=-1,\ l=-2,\ k=j=|\alpha|=0$

By (3.1), we get

$$\text{case b)}=\sum_{|\beta'|=1}\sum_{|\delta'|=1}^{2}\frac{(-i)^{2+|\delta'|}}{\delta'!}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\partial_{x'}^{\beta'}f_1\partial_{x'}^{\delta'}f_2$$

$$\times\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi'}^{\beta'}\pi_{\xi_n}^{+}\sigma_L^F\times\partial_{\xi'}^{\delta'}\partial_{\xi_n}\sigma_{-2+|\delta'|}^F](x_0)d\xi_n\sigma(\xi')d^{n-1}x'. \quad (3.20)$$

Similar to the discussions in case b), case c) also has the form in Theorem 3.1, so we proved Theorem 3.1. □

4 Some Remarks

In this section, we will point out the way of computations of a_{ij} in Theorem 3.1 and $\Omega_3(f_1,f_2)$ in the case of f_1,f_2 depending on x_n by some remarks.

Remark 1 Since the computation of $\pi_{\xi_n}^{+}\sigma_{-1}^F(x_0)$ is a little tedious, the computation of case c) is more direct than the computation of case b). So we try to use the computation of case c) and some simple computations instead of the computation of case b). By the Leibniz rule, trace property and "++" and "- -" vanishing after the integration over ξ_n (for details, see [FGLS]), then

$$\int_{-\infty}^{+\infty}\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}^F(\xi',\xi_n)\times\sigma_L^F(\eta',\xi_n)](x_0)d\xi_n$$

$$=\int_{-\infty}^{+\infty}\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi_n}\sigma_{-1}^F(\xi',\xi_n)\times\sigma_L^F(\eta',\xi_n)](x_0)d\xi_n$$

$$-\int_{-\infty}^{+\infty}\mathrm{trace}_{\wedge^2T^*M}[\partial_{\xi_n}\pi_{\xi_n}^{-}\sigma_{-1}^F(\xi',\xi_n)\times\sigma_L^F(\eta',\xi_n)](x_0)d\xi_n$$

$$
\begin{aligned}
&= -\int_{-\infty}^{+\infty} \mathrm{trace}_{\wedge^2 T^*M}[\sigma^F_{-1}(\xi', \xi_n) \times \partial_{\xi_n}\sigma^F_L(\eta', \xi_n)](x_0)d\xi_n \\
&\quad -\int_{-\infty}^{+\infty} \mathrm{trace}_{\wedge^2 T^*M}[\pi^+_{\xi_n}\sigma^F_L(\eta', \xi_n) \times \partial_{\xi_n}\pi^-_{\xi_n}\sigma^F_{-1}(\xi', \xi_n)](x_0)d\xi_n \\
&= -\int_{-\infty}^{+\infty} \mathrm{trace}_{\wedge^2 T^*M}[\sigma^F_{-1}(\xi', \xi_n) \times \partial_{\xi_n}\sigma^F_L(\eta', \xi_n)](x_0)d\xi_n \\
&\quad -\int_{-\infty}^{+\infty} \mathrm{trace}_{\wedge^2 T^*M}[\pi^+_{\xi_n}\sigma^F_L(\eta', \xi_n) \times \partial_{\xi_n}\sigma^F_{-1}(\xi', \xi_n)](x_0)d\xi_n.
\end{aligned}
$$

For computations of case a) II) and III), we have a similar remark. But we may not get the sum of case b) and case c) is zero through the above computations although we conjecture that it should vanish and $\Omega_3(f_1, f_2)$ is also zero.

Remark 2 The computations of the trace of some operators will appear in this case. We just compute an example and the others are similar. In the following, we compute the equality:

$$\mathrm{trace}_{\wedge^2 T^*M}\{[\partial_{x_n}p(\xi)]p(\eta)\}(x_0) = h'(0)[a_{n,m}\langle\xi', \eta'\rangle^2 + b_{n,m}|\xi'|^2|\eta|^2](x_0) + 8h'(0)\xi_n\eta_n\langle\xi', \eta'\rangle, \tag{4.1}$$

where $C^m_n - a_{n,m} = b_{n,m} = C^{m-2}_{n-2} + C^m_{n-2} - 2C^{m-1}_{n-2}$ and $C^m_n = \frac{n!}{m!(n-m)!}$.

Proof. By (3.5),

$$\partial_{x_n}p(\xi)(x_0) = h'(0)[\varepsilon(\xi)\iota(\xi') - \iota(\xi')\varepsilon(\xi)](x_0) = h'(0)p(\xi', 0) + \xi_n B, \tag{4.2}$$

where $B = h'(0)[\varepsilon(dx_n)\iota(\xi') - \iota(\xi')\varepsilon(dx_n)](x_0)$. By the well-known equality

$$\varepsilon_{m-1}(\xi)\iota_m(\eta) + \iota_{m+1}(\eta)\varepsilon_m(\xi) = \langle\xi, \eta\rangle I_m, \tag{4.3}$$

then

$$\varepsilon(dx_n)\iota(\xi') - \iota(\xi')\varepsilon(dx_n) = 2\varepsilon(dx_n)\iota(\xi');\ p(\eta) = 2\varepsilon(\eta)\iota(\eta) - \langle\eta, \eta\rangle I_m. \tag{4.4}$$

So by (4.2), (4.4) and Theorem 4.3 in [U],

$$
\begin{aligned}
\mathrm{trace}_{\wedge^2 T^*M}\{[\partial_{x_n}p(\xi)]p(\eta)\}(x_0) &= h'(0)[a_{n,m}\langle\xi', \eta'\rangle^2 + b_{n,m}|\xi'|^2|\eta|^2](x_0) \\
+ 4h'(0)\xi_n \mathrm{trace}_{\wedge^2 T^*M}[\varepsilon(dx_n)\iota(\xi')\varepsilon(\eta)\iota(\eta)] &- 2|\eta|^2 h'(0)\xi_n \mathrm{trace}_{\wedge^2 T^*M}[\varepsilon(dx_n)\iota(\xi')]
\end{aligned}
\tag{4.5}
$$

By (4.3) and the trace property, we have

$$\mathrm{trace}_{\wedge^2 T^*M}[\varepsilon(dx_n)\iota(\xi')] = 0. \tag{4.6}$$

As in [U,p.12-13], we write

$$a_m(\xi_1, \xi_2, \eta_1, \eta_2) = \mathrm{trace}_{\wedge^m T^*M}[\varepsilon_{m-1}(\xi_1)\iota_m(\xi_2)\varepsilon_{m-1}(\eta_1)\iota_m(\eta_2)].$$

then

$$a_{m+1}(\eta_1,\xi_2,\xi_1,\eta_2)=a_m(\xi_1,\xi_2,\eta_1,\eta_2)+\langle\xi_1,\xi_2\rangle\langle\eta_1,\eta_2\rangle[2A_{n,m}-C_n^m], \tag{4.7}$$

where $A_{n,m}=C_n^m-C_n^{m-1}+\cdots+(-1)^mC_n^0$. So $a_1(\xi_1,\xi_2,\eta_1,\eta_2)=\langle\eta_2,\xi_1\rangle\langle\xi_2,\eta_1\rangle$ and

$$a_2(\eta_1,\xi_2,\xi_1,\eta_2)=\langle\eta_2,\xi_1\rangle\langle\xi_2,\eta_1\rangle+\langle\xi_1,\xi_2\rangle\langle\eta_1,\eta_2\rangle[2A_{n,1}-C_n^1]. \tag{4.8}$$

So by (4.8) and $n=4$

$$\mathrm{trace}_{\wedge^2T^*M}[\varepsilon(dx_n)\iota(\xi')\varepsilon(\eta)\iota(\eta)]=a_2(dx^n,\xi',\eta,\eta)=2\eta_n\langle\xi',\eta'\rangle. \tag{4.9}$$

By (4.5),(4.6) and (4.9), we prove the equality (4.1). □

Remark 3 When $n=4$ and f_1,f_2 depend on x_n, by (2.6) and considering the sum is taken over $-(r+l)+|\alpha|+k+j=3,\ \ r,l\leq -1,\ \ 1\leq|\beta|=|\beta'|+\beta''\leq -r,\ \ 1\leq|\delta|=|\delta'|+\delta''\leq -l$, similar to Section 3, we compute $\Omega_{n-1}(f_1,f_2)(x_0)$ as the sum of 24 cases about $(r,l,k,j,\alpha,\beta',\beta'',\delta',\delta'')$. This can not add to new technical difficulties except for a little tedious computations.

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