ANOMALY CANCELLATION AND MODULARITY II: THE $E_8 \times E_8$ CASE

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Abstract. In this paper we show that both of the Green-Schwarz anomaly factorization formula for the gauge group $E_8 \times E_8$ and the Hořava-Witten anomaly factorization formula for the gauge group E_8 can be derived through modular forms of weight 14. This answers a question of J. H. Schwarz. We also establish generalizations of these factorization formulas and obtain a new Hořava-Witten type factorization formula.

Introduction

In [15], [8] and [9], it has been shown that both of the Alvarez-Gaumé-Witten miraculous anomaly cancellation formula [2] and the Green-Schwarz anomaly factorization formula [7] for the gauge group $SO(32)$ can be derived (and extended) through a pair of modularly related modular forms, which are over the modular subgroup $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively. In answering a question of J. H. Schwarz [16], we deal with the remaining case of gauge group $E_8 \times E_8$ in this article.

Let $Z \to X \to B$ be a fiber bundle with fiber Z being 10 dimensional. Let TZ be the vertical tangent bundle equipped with a metric g^{TZ} and an associated Levi-Civita connection ∇^{TZ} (cf. [3, Proposition 10.2]). Let $R^{TZ} = (\nabla^{TZ})^2$ be the curvature of ∇^{TZ}, which we also for simplicity denote by R. Let $T_{\mathbf{C}}Z$ be the complexification of TZ with the induced Hermitian connection $\nabla^{T_{\mathbf{C}}Z}$.

Let $(P_1, \vartheta_1), (P_2, \vartheta_2)$ be two principal E_8 bundles with connections over X. Let ρ be the adjoint representation of E_8. Let $W_i = P_i \times_\rho \mathbf{C}^{248},\ i = 1, 2$ be the associated vector bundles, which are of rank 248. We equip both W_1, W_2 with Hermitian metrics and Hermitian connections respectively. Let F_i denote the curvature of the bundle W_i. Let "Tr" denote the trace in the adjoint representation. Then one has $\mathrm{Tr}F_i^{2n+1} = 0$ (cf. the proof of Theorem 2.1 in this article), $\mathrm{Tr}F_i^4 = \frac{1}{100}(\mathrm{Tr}F_i^2)^2, \mathrm{Tr}F_i^6 = \frac{1}{7200}(\mathrm{Tr}F_i^2)^3$ (cf. [1]). It's easy to see that $c_2(W_i) = -\frac{1}{2}\mathrm{Tr}F_i^2$. Simply denote $\mathrm{Tr}F_1^n + \mathrm{Tr}F_2^n$ by $\mathrm{Tr}F^n$.

The Green-Schwarz anomaly formula [7] asserts that the following factorization for the 12 forms holds,[1]

(0.1)

$$\begin{aligned}
&I_{12}\\
=&\left\{\widehat{A}(TZ)\mathrm{ch}(W_1+W_2)+\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)-2\widehat{A}(TZ)\right\}^{(12)}\\
=&\frac{-1}{64\pi^6}\frac{1}{720}\left(-\frac{15}{8}\mathrm{tr}R^2\mathrm{tr}R^4-\frac{15}{32}(\mathrm{tr}R^2)^3+\mathrm{Tr}F^6+\mathrm{Tr}F^2\left(\frac{1}{16}\mathrm{tr}R^4+\frac{5}{64}(\mathrm{tr}R^2)^2\right)-\frac{5}{8}\mathrm{Tr}F^4\mathrm{tr}R^2\right)\\
=&\frac{-1}{4\pi^2}\frac{1}{2}\left(\mathrm{tr}R^2-\frac{1}{30}\mathrm{Tr}F^2\right)\cdot\frac{1}{16\pi^4}\frac{1}{180}\left(\frac{1}{960}(\mathrm{Tr}F^2)^2-\frac{5}{16}\mathrm{Tr}F^4+\frac{1}{32}\mathrm{tr}R^2\mathrm{Tr}F^2-\frac{15}{16}\mathrm{tr}R^4-\frac{15}{64}(\mathrm{tr}R^2)^2\right)\\
&\cdot\\
=&\left(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2))\right)\cdot I_8.
\end{aligned}$$

[1] In what follows, we will write characteristic forms without specifying the connections when there is no confusion (cf. [17]).

In [11] and [12], Hořava and Witten observed, on the other hand, that the following anomaly factorization formula holds for each $i = 1, 2$,

(0.2)

$$\begin{aligned}
\widehat{I}^i_{12} &= \left\{ \widehat{A}(TZ)\mathrm{ch}(W_i) + \frac{1}{2}\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z) - \widehat{A}(TZ) \right\}^{(12)} \\
&= \frac{-1}{64\pi^6}\frac{1}{1440}\left(-\frac{15}{8}\mathrm{tr}R^2\mathrm{tr}R^4 - \frac{15}{32}(\mathrm{tr}R^2)^3 + 2\mathrm{Tr}F_i^6 + \mathrm{Tr}F_i^2\left(\frac{1}{8}\mathrm{tr}R^4 + \frac{5}{32}(\mathrm{tr}R^2)^2\right) - \frac{5}{4}\mathrm{Tr}F_i^4\mathrm{tr}R^2 \right) \\
&= \frac{-1}{4\pi^2}\frac{1}{4}\left(\mathrm{tr}R^2 - \frac{1}{15}\mathrm{Tr}F_i^2\right)\cdot \widehat{I}^i_8 \\
&= \left(\frac{1}{2}p_1(TZ) + \frac{1}{30}c_2(W_i)\right)\cdot \widehat{I}^i_8,
\end{aligned}$$

where $\widehat{I}^i_8$ can be written explicitly as

$$\widehat{I}^i_8 = \frac{1}{16\pi^4}\frac{1}{24}\left(-\frac{1}{4}\left(\frac{1}{2}\mathrm{tr}R^2 - \frac{1}{30}\mathrm{Tr}F_i^2\right)^2 - \frac{1}{8}\mathrm{tr}R^4 + \frac{1}{32}(\mathrm{tr}R^2)^2 \right),$$

and therefore

$$I_{12} = \widehat{I}^1_{12} + \widehat{I}^2_{12} = \left(\frac{1}{2}p_1(TZ) + \frac{1}{30}c_2(W_1)\right)\cdot \widehat{I}^1_8 + \left(\frac{1}{2}p_1(TZ) + \frac{1}{30}c_2(W_2)\right)\cdot \widehat{I}^2_8.$$

The purpose of this article is to show that the above anomaly factorization formulas can also be derived natually from modularity as in the orthogonal group case dealt with in [9]. This provides a positive answer to a question of J. H. Schwarz mentioned at the beginning of the article.

To be more precse, we will construct in Section 2 a modular form $\mathcal{Q}(P_i, P_j, \tau)$ of weight 14 over $SL(2, \mathbf{Z})$, for any $i,\ j \in \{1,\ 2\}$, such that when $i = 1, j = 2$, the modularity of $\mathcal{Q}(P_1, P_2, \tau)$ gives the Green-Schwarz factorization formula (0.1), while when $i = j$, the modularity of $\mathcal{Q}(P_i, P_i, \tau)$ gives the Hořava-Witten factorization formula (0.2). Actually what we construct is a more general modular form $\mathcal{Q}(P_i, P_j, \xi, \tau)$, which involves a complex line bundle (or equivalently a rank two real oriented bundle) and we are able to obtain generalizations of the Green-Schwarz formula and the Hořava-Witten formula by using the associated modularity. Our construction of the modular form $\mathcal{Q}(P_i, P_j, \xi, \tau)$ involves the basic representation of the affine Kac-Moody algebra of E_8.

Inspired by our modular method of deriving the Green-Schwarz and Hořava-Witten factorization formulas, we also construct a modular form $\mathcal{R}(P_i, \xi, \tau)$ of weight 10 over $SL(2, \mathbf{Z})$, the modularity of which will give us a new factorization formula of Hořava-Witten type. See Theorem 0.2 for details. It would be interesting to compare (0.8), (0.9) with the Hořava-Witten factorization (0.2) or (0.6). Actually another interesting question of J.H. Schwarz is to construct quantum field theories associated to the generalized anomaly factorization formulas in this paper and [9].

In the rest of this section, we will present our generalized Green-Schwarz and Hořava-Witten formula, as well as the new formulas of Hořava-Witten type obtained from $\mathcal{R}(P_i, \xi, \tau)$. They will be proved in Section 2 by using modularity after briefly reviewing some knowledge of the affine Kac-Moody algebra of E_8 in Section 1.

Let ξ be a rank two real oriented Euclidean vector bundle over X carrying a Euclidean connection ∇^ξ. Let $c = e(\xi, \nabla^\xi)$ be the Euler form canonically associated to ∇^ξ (cf. [17, Section 3.4]).

Theorem 0.1. *The following identities hold,*

(0.3)

$$\Big\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(W_1+W_2)+\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(T_{\mathbf{C}}Z)-2\widehat{A}(TZ)e^{\frac{c}{2}}+\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}\Big\}^{(12)}$$

$$=\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_1)+c_2(W_2))\right)$$

$$\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_1)+c_2(W_2)\right)}-1}{p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_1)+c_2(W_2))}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\mathfrak{A})+e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_1)+c_2(W_2)\right)}\widehat{A}(TZ)e^{\frac{c}{2}}\right\}^{(8)},$$

where $\mathfrak{A}=W_1+W_2+T_{\mathbf{C}}Z-2+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}$;
and for each i,

(0.4)

$$\left\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(W_i)+\frac{1}{2}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(T_{\mathbf{C}}Z)-\widehat{A}(TZ)e^{\frac{c}{2}}+\frac{1}{2}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\right\}^{(12)}$$

$$=\left(\frac{1}{2}p_1(TZ)-\frac{3}{2}c^2+\frac{1}{30}c_2(W_i)\right)$$

$$\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{15}c_2(W_i)\right)}-1}{p_1(TZ)-3c^2+\frac{1}{15}c_2(W_i)}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\mathfrak{B}_i)+e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{15}c_2(W_i)\right)}\widehat{A}(TZ)e^{\frac{c}{2}}\right\}^{(8)},$$

where $\mathfrak{B}_i=2W_i+T_{\mathbf{C}}Z-2+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}$.

If ξ is trivial, we obtain the Green-Schwarz formula (0.1) for $E_8\times E_8$ and the Hořava-Witten formula (0.2) for E_8 in the following corollary.

Corollary 0.1. *One has*

(0.5)

$$\left\{\widehat{A}(TZ)\mathrm{ch}(W_1+W_2)+\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)-2\widehat{A}(TZ)\right\}^{(12)}$$

$$=\left(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2))\right)$$

$$\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2)\right)}-1}{p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2))}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{C})+e^{\frac{1}{24}\left(p_1(TZ)+\frac{1}{30}(c_2(W_1)+c_2(W_2)\right)}\widehat{A}(TZ)\right\}^{(8)},$$

where $\mathfrak{C}=W_1+W_2+T_{\mathbf{C}}Z-2$;
and for each i,

$$\left\{\widehat{A}(TZ)\mathrm{ch}(W_i)+\frac{1}{2}\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)-\widehat{A}(TZ)\right\}^{(12)}$$

$$=\left(\frac{1}{2}p_1(TZ)+\frac{1}{30}c_2(W_i)\right) \tag{0.6}$$

$$\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)+\frac{1}{15}c_2(W_i)\right)}-1}{p_1(TZ)+\frac{1}{15}c_2(W_i)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{D}_i)+e^{\frac{1}{24}\left(p_1(TZ)+\frac{1}{15}c_2(W_i)\right)}\widehat{A}(TZ)\right\}^{(8)},$$

where $\mathfrak{D}_i=2W_i+T_{\mathbf{C}}Z-2$.

Remark 0.1. *It can be checked by direct computation that the second factors in the right hand sides of (0.5) and (0.6) are equal to* I_8 *and* $\widehat{I_8^i}$ *respectively.*

We now state a new factorization formula, which is of the Hořava-Witten type.

Theorem 0.2. *For each i, the following identity holds,*

(0.7)

$$\begin{aligned}&\left\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(W_i)+\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(T_{\mathbf{C}}Z)+246\widehat{A}(TZ)e^{\frac{c}{2}}+\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\right\}^{(12)}\\&=\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\\&\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}-1}{p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\mathfrak{E}_i)+e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)e^{\frac{c}{2}}\right\}^{(8)},\end{aligned}$$

where $\mathfrak{E}_i=W_i+T_{\mathbf{C}}Z+246+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}$;
if ξ is trivial, we have

$$\begin{aligned}&\left\{\widehat{A}(TZ)\mathrm{ch}(W_i)+\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)+246\widehat{A}(TZ)\right\}^{(12)}\\&=\left(p_1(TZ)+\frac{1}{30}c_2(W_i)\right)\\&\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)+\frac{1}{30}c_2(W_i)\right)}-1}{p_1(TZ)+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{F}_i)+e^{\frac{1}{24}\left(p_1(TZ)+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\right\}^{(8)},\end{aligned}\tag{0.8}$$

where $\mathfrak{F}_i=W_i+T_{\mathbf{C}}Z+246$.

Remark 0.2. *We can express (0.8) by direct computations as follows,*

(0.9)

$$\begin{aligned}&\frac{-1}{64\pi^6}\frac{1}{1440}\left(-\frac{15}{4}\mathrm{tr}R^2\mathrm{tr}R^4-\frac{15}{16}(\mathrm{tr}R^2)^3+2\mathrm{Tr}F_i^6+\mathrm{Tr}F_i^2\left(\frac{1}{8}\mathrm{tr}R^4+\frac{5}{32}(\mathrm{tr}R^2)^2\right)-\frac{5}{4}\mathrm{Tr}F_i^4\mathrm{tr}R^2\right)\\=&\frac{-1}{4\pi^2}\frac{1}{2}\left(\mathrm{tr}R^2-\frac{1}{30}\mathrm{Tr}F_i^2\right)\cdot\frac{1}{16\pi^4}\frac{1}{180}\left(-\frac{1}{480}(\mathrm{Tr}F_i^2)^2+\frac{1}{32}\mathrm{tr}R^2\mathrm{Tr}F_i^2-\frac{15}{16}\mathrm{tr}R^4-\frac{15}{64}(\mathrm{tr}R^2)^2\right)\\=&\left(p_1(TZ)+\frac{1}{30}c_2(W_i)\right)\cdot\widehat{J}_8^i.\end{aligned}$$

Remark 0.3. *As in* [16], *one may ask whether there is a phyics model corresponding to (0.8) and (0.9).*

1. The Basic Representation of Affine E_8

In this section we briefly review the basic representation theory for the affine E_8 by following [13] (see also [14]).

Let $\mathfrak{g}$ be the Lie algebra of E_8. Let $\langle,\rangle$ be the Killing form on $\mathfrak{g}$. Let $\widetilde{\mathfrak{g}}$ be the affine Lie algebra corresponding to $\mathfrak{g}$ defined by

$$\widetilde{\mathfrak{g}}=\mathbf{C}[t,t^{-1}]\otimes\mathfrak{g}\oplus\mathbf{C}c,$$

with bracket

$$[P(t)\otimes x+\lambda c,Q(t)\otimes y+\mu c]=P(t)Q(t)\otimes[x,y]+\langle x,y\rangle\,\mathrm{Res}_{t=0}\left(\frac{dP(t)}{dt}Q(t)\right)c.$$

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra obtained from $\widetilde{\mathfrak{g}}$ by adding a derivation $t\frac{d}{dt}$ which operates on $\mathbf{C}[t,t^{-1}]\otimes\mathfrak{g}$ in an obvious way and sends c to 0.

The basic representation $V(\Lambda_0)$ is the $\widehat{\mathfrak{g}}$-module defined by the property that there is a nonzero vector v_0 (highest weight vector) in $V(\Lambda_0)$ such that $cv_0 = v_0, (\mathbf{C}[t] \oplus \mathbf{C}t\frac{d}{dt})v_0 = 0$. Setting $V_k := \{v \in V(\Lambda_0) | t\frac{d}{dt} = -kv\}$ gives a $\mathbf{Z}_+$-gradation by finite spaces. Since $[g, d] = 0$, each V_k is a representation of $\mathfrak{g}$. Moreover, V_1 is the adjoint representation of E_8.

Let $q = e^{2\pi\sqrt{-1}\tau}$. Fix a basis $\{z_i\}_{i=1}^8$ for the Cartan subalgebra. The character of the basic representation is given by

$$\mathrm{ch}(z_1, z_2, \cdots, z_8, \tau) := \sum_{k=0}^{\infty} (\mathrm{ch}V_k)(z_1, z_2, \cdots, z_8) q^k = \varphi(\tau)^{-r} \Theta_{\mathfrak{g}}(z_1, z_2, \cdots, z_8, \tau),$$

where $\varphi(\tau) = \prod_{n=1}^{\infty}(1-q^n)$ so that $\eta(\tau) = q^{1/24}\varphi(\tau)$ is the Dedekind η function; $\Theta_{\mathfrak{g}}(z_1, z_2, \cdots, z_8, \tau)$ is the theta function defined on the root lattice Q by

$$\Theta_{\mathfrak{g}}(z_1, z_2, \cdots, z_8, \tau) = \sum_{\gamma \in Q} q^{|\gamma|^2/2} e^{2\pi\sqrt{-1}\gamma(\overrightarrow{z})}.$$

It is proved in [6] (cf. [10]) that there is a basis for the E_8 root lattice such that

$$\Theta_{\mathfrak{g}}(z_1, \cdots .z_8, \tau) = \frac{1}{2}\left(\prod_{l=1}^{8} \theta(z_l, \tau) + \prod_{l=1}^{8} \theta_1(z_l, \tau) + \prod_{l=1}^{8} \theta_2(z_l, \tau) + \prod_{l=1}^{8} \theta_3(z_l, \tau)\right), \tag{1.1}$$

where θ and θ_i $(i = 1,\ 2,\ 3)$ are the Jacobi theta functions (cf. [4] and [8]).

2. Derivation of Green-Schwarz and Horava-Witten type anomaly factorizations via modularity

In this section, we will derive the Green-Schwarz and Hořava-Witten type factorization formulas in Theorems 0.1 and 0.2 via modularity.

For the principal E_8 bundles $P_i,\ i = 1, 2$, consider the associated bundles

$$\mathcal{V}_i = \sum_{k=0}^{\infty} (P_i \times_{\rho_k} V_k)\, q^k \in K(X)[[q]].$$

Since ρ_1 is the adjoint representation of E_8, we have $W_i = P_i \times_{\rho_1} V_1$.

Following [5], set

$$\Theta(T_{\mathbf{C}}Z, \xi_{\mathbf{C}}) := \left(\bigotimes_{m=1}^{\infty} S_{q^m}(\widetilde{T_{\mathbf{C}}Z})\right) \otimes \left(\bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\widetilde{\xi_{\mathbf{C}}})\right) \otimes \left(\bigotimes_{u=1}^{\infty} \Lambda_{-q^{u-1/2}}(\widetilde{\xi_{\mathbf{C}}})\right) \otimes \left(\bigotimes_{v=1}^{\infty} \Lambda_{q^{v-1/2}}(\widetilde{\xi_{\mathbf{C}}})\right) \in K(X)[[q]],$$

where $\xi_{\mathbf{C}}$ is the complexification of ξ, and for a complex vector bundle E, $\widetilde{E} := E - \mathbf{C}^{\mathrm{rk}(E)}$.

Clearly, $\Theta(T_{\mathbf{C}}Z, \xi_{\mathbf{C}})$ admits a formal Fourier expansion in q as

$$\Theta(T_{\mathbf{C}}Z, \xi_{\mathbf{C}}) = \mathbf{C} + B_1 q + B_2 q^2 \cdots, \tag{2.1}$$

where the B_j's are elements in the semi-group formally generated by complex vector bundles over X. Moreover, they carry canonically induced connections denoted by ∇^{B_j}. Let ∇^{Θ} be the induced connection with q-coefficients on Θ.

For $1 \le i, j \le 2$, set

(2.2)

$$\mathcal{Q}(P_i, P_j, \xi, \tau)$$

$$:= \left\{ e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j))\right)} \widehat{A}(TZ) \cosh\left(\frac{c}{2}\right) \mathrm{ch}\left(\Theta(T_{\mathbf{C}}Z, \xi_{\mathbf{C}})\right) \varphi(\tau)^{16} \mathrm{ch}(\mathcal{V}_i)\mathrm{ch}(\mathcal{V}_j) \right\}^{(12)}.$$

Theorem 2.1. *$\mathcal{Q}(P_i, P_j, \xi, \tau)$ is a modular form of weight 14 over $SL(2, \mathbf{Z})$.*

Proof: By the knowledge reviewed in Section 2, we see that there are formal two forms $y_l^i, 1 \leq l \leq 8, i = 1, 2$ such that

$$\varphi(\tau)^8 \mathrm{ch}(\mathcal{V}_i) = \frac{1}{2}\left(\prod_{l=1}^{8}\theta(y_l^i,\tau) + \prod_{l=1}^{8}\theta_1(y_l^i,\tau) + \prod_{l=1}^{8}\theta_2(y_l^i,\tau) + \prod_{l=1}^{8}\theta_3(y_l^i,\tau)\right). \tag{2.3}$$

Since $\theta(z,\tau)$ is an odd function about z and we only take forms of degrees not greater than12, one has

$$\varphi(\tau)^8 \mathrm{ch}(\mathcal{V}_i) = \frac{1}{2}\left(\prod_{l=1}^{8}\theta_1(y_l^i,\tau) + \prod_{l=1}^{8}\theta_2(y_l^i,\tau) + \prod_{l=1}^{8}\theta_3(y_l^i,\tau)\right). \tag{2.4}$$

Since $\theta_1(z,\tau), \theta_2(z,\tau)$ and $\theta_3(z,\tau)$ are all even functions about z, the right hand side of the above equality only contains even powers of y_j^i's. Therefore $\mathrm{ch}(W_i)$ only consists of forms of degrees divisible by 4. So

$$\mathrm{ch}(\mathcal{V}_i) = 1 + \mathrm{ch}(W_i)q + \cdots = 1 + (248 - c_2(W_i) + \cdots)q + \cdots. \tag{2.5}$$

On the other hand,

$$\frac{1}{2}\left(\prod_{l=1}^{8}\theta_1(y_l^i,\tau) + \prod_{l=1}^{8}\theta_2(y_l^i,\tau) + \prod_{l=1}^{8}\theta_3(y_l^i,\tau)\right) = 1 + \left(240 + 30\sum_{l=1}^{8}(y_l^i)^2 + \cdots\right)q + O(q^2). \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have

$$\sum_{l=1}^{8}\left(y_l^i\right)^2 = -\frac{1}{30}c_2(W_i). \tag{2.7}$$

Let $\{\pm 2\pi\sqrt{-1}x_l\}$ be the formal Chern roots for $(TZ_{\mathbf{C}}, \nabla^{TZ_{\mathbf{C}}})$. Let $c = 2\pi\sqrt{-1}u$. One has

(2.8)

$$\begin{aligned}
&\mathcal{Q}(P_i, P_j, \xi, \tau) \\
&= \left\{e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta(T_{\mathbf{C}}Z, \xi_{\mathbf{C}})\right)\varphi(\tau)^{16}\mathrm{ch}(\mathcal{V}_i)\mathrm{ch}(\mathcal{V}_j)\right\}^{(12)} \\
&= \left\{e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\left(\prod_{l=1}^{5}\left(x_l\frac{\theta'(0,\tau)}{\theta(x_l,\tau)}\right)\right)\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\right. \\
&\quad \left.\cdot\frac{1}{4}\left(\prod_{l=1}^{8}\theta_1(y_l^i,\tau) + \prod_{l=1}^{8}\theta_2(y_l^i,\tau) + \prod_{l=1}^{8}\theta_3(y_l^i,\tau)\right)\left(\prod_{l=1}^{8}\theta_1(y_l^j,\tau) + \prod_{l=1}^{8}\theta_2(y_l^j,\tau) + \prod_{l=1}^{8}\theta_3(y_l^j,\tau)\right)\right\}^{(12)}.
\end{aligned}$$

Then we can preform the transformation formulas for the theta functions and $E_2(\tau)$ (c.f. [4] and [8]) to show that $\mathcal{Q}(P_i, P_j, \xi, \tau)$ is a modular form of weight 14 over $SL(2, \mathbf{Z})$. Q.E.D.

Proof of Theorem 0.1: Expanding the q-series, we have

(2.9)

$$
\begin{aligned}
& e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta(T_{\mathbf{C}}Z,\xi_{\mathbf{C}})\right)\varphi(\tau)^{16}\mathrm{ch}(\mathcal{V}_i)\mathrm{ch}(\mathcal{V}_j)\\
=& \Big(e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\\
&\quad -e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)q+O(q^2)\Big)\\
&\cdot\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\mathbf{C}+B_1q+O(q^2))(1-16q+O(q^2))(1+\mathrm{ch}(W_i)q+O(q^2))(1+\mathrm{ch}(W_j)q+O(q^2))\\
=& e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\\
&+q\Big(e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-16+W_i+W_j)\\
&\quad -e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\Big)\\
&+O(q^2).
\end{aligned}
$$

It is well known that modular forms over $SL(2,\mathbf{Z})$ can be expressed as polynomials of the Eisenstein series $E_4(\tau)$, $E_6(\tau)$, where

$$E_4(\tau)=1+240q+2160q^2+6720q^3+\cdots, \tag{2.10}$$

$$E_6(\tau)=1-504q-16632q^2-122976q^3+\cdots. \tag{2.11}$$

Their weights are 4 and 6 respectively.

Since the weight of the modular form $\mathcal{Q}(P_i,P_j,\xi,\tau)$ is 14, it must be a multiple of

$$E_4(\tau)^2E_6(\tau)=1-24q+\cdots. \tag{2.12}$$

So from (2.9) and (2.12), we have

(2.13)

$$
\begin{aligned}
&\left\{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-16+W_i+W_j)\right\}^{(12)}\\
&\quad -\left\{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}\\
=&-24\left\{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+B_1+8)\right\}^{(12)}\\
=&\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\\
&\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)\right)}-1}{p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+B_1+8)\right.\\
&\quad\left.+e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(8)}.
\end{aligned}
\tag{2.14}
$$

To find B_1, we have

$$
\begin{aligned}
&\Theta(T_{\mathbf{C}}Z,\xi_{\mathbf{C}})\\
=&\left(\bigotimes_{m=1}^{\infty}S_{q^m}(\widetilde{T_{\mathbf{C}}Z})\right)\otimes\left(\bigotimes_{n=1}^{\infty}\Lambda_{q^n}(\widetilde{\xi_{\mathbf{C}}})\right)\otimes\left(\bigotimes_{u=1}^{\infty}\Lambda_{-q^{u-1/2}}(\widetilde{\xi_{\mathbf{C}}})\right)\otimes\left(\bigotimes_{v=1}^{\infty}\Lambda_{q^{v-1/2}}(\widetilde{\xi_{\mathbf{C}}})\right)\\
=&(1+(T_{\mathbf{C}}Z-10)q+O(q^2))\otimes(1+\widetilde{\xi_{\mathbf{C}}}q+O(q^2))\\
&\otimes(1-\widetilde{\xi_{\mathbf{C}}}q^{1/2}-2\widetilde{\xi_{\mathbf{C}}}q+O(q^{3/2}))\otimes(1+\widetilde{\xi_{\mathbf{C}}}q^{1/2}-2\widetilde{\xi_{\mathbf{C}}}q+O(q^{3/2}))\\
=&1+(T_{\mathbf{C}}Z-10+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})q+O(q^2).
\end{aligned}
\tag{2.15}
$$

So $B_1=T_{\mathbf{C}}Z-10+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}$.

Plugging B_1 into (2.14), we have

(2.16)

$$
\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+T_{\mathbf{C}}Z-2+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\right\}^{(12)}\\
=&\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)\\
&\cdot\left\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}-1}{p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+W_j+T_{\mathbf{C}}Z-2+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\right.\\
&\left.\quad+e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}(c_2(W_i)+c_2(W_j))\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(8)}.
\end{aligned}
$$

Since $\mathrm{ch}(W_i),\mathrm{ch}(W_j)$ only contribute degree $4l$ forms, we can replace $\cosh\left(\frac{c}{2}\right)$ by $e^{\frac{c}{2}}$. Then in (2.16), putting $i=1,\ j=2$ gives (0.4) and putting $i=j$ gives (0.5). Q.E.D.

To prove theorem 0.2, for each i, set

$$
\begin{aligned}
&\mathcal{R}(P_i,\xi,\tau)\\
:=&\left\{e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta(T_{\mathbf{C}}Z,\xi_{\mathbf{C}})\right)\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i)\right\}^{(12)}.
\end{aligned}
\tag{2.17}
$$

Theorem 2.2. *$\mathcal{R}(P_i,\xi,\tau)$ is a modular form of weight 10 over $SL(2,\mathbf{Z})$.*

Proof: This can be similarly proved as Theorem 2.1 by seeing that

$$
\begin{aligned}
&\mathcal{R}(P_i,\xi,\tau)\\
=&\left\{e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta(T_{\mathbf{C}}Z,\xi_{\mathbf{C}})\right)\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i)\right\}^{(12)}\\
=&\left\{e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\left(\prod_{l=1}^{5}\left(x_l\frac{\theta'(0,\tau)}{\theta(x_l,\tau)}\right)\right)\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\right.\\
&\left.\cdot\frac{1}{2}\left(\prod_{l=1}^{8}\theta_1(y_l^i,\tau)+\prod_{l=1}^{8}\theta_2(y_l^i,\tau)+\prod_{l=1}^{8}\theta_3(y_l^i,\tau)\right)\right\}^{(12)},
\end{aligned}
\tag{2.18}
$$

and then apply the transformation laws of theta functions. Q.E.D.

Proof of Theorem 0.2: Similar as in the proof of Theorem 0.1, expanding the q-series, we have

$$
\begin{aligned}
& e^{\frac{1}{24}E_2(\tau)\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta(T_{\mathbf{C}}Z,\xi_{\mathbf{C}})\right)\varphi(\tau)^8\mathrm{ch}(\mathcal{V}_i)\\
=&\Big(e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\\
&\quad -e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)q+O(q^2)\Big)\\
&\cdot\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(\mathbf{C}+B_1q+O(q^2))(1-8q+O(q^2))(1+\mathrm{ch}(W_i)q+O(q^2))\\
=&e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\\
&+q\Big(e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-8+W_i)\\
&\quad -e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\Big)\\
&+O(q^2).
\end{aligned}
\tag{2.19}
$$

However modular form of weight 10 must be a multiple of $E_4(\tau)E_6(\tau)=1-264q+\cdots$, so we have

$$
\begin{aligned}
&\left\{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1-8+W_i)\right\}^{(12)}\\
&\quad-\left\{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}\\
=&-264\left\{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\right\}^{(12)}.
\end{aligned}
\tag{2.20}
$$

Therefore

$$
\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+B_1+256)\right\}^{(12)}\\
=&\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\\
&\cdot\Bigg\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}-1}{p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+B_1+256)\\
&\quad+e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\Bigg\}^{(8)}.
\end{aligned}
\tag{2.21}
$$

Plugging in B_1, we have

$$
\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+T_{\mathbf{C}}Z+246+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\right\}^{(12)}\\
=&\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)\\
&\cdot\Bigg\{-\frac{e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}-1}{p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(W_i+T_{\mathbf{C}}Z+246+\widetilde{\xi_{\mathbf{C}}}+3\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\\
&\quad+e^{\frac{1}{24}\left(p_1(TZ)-3c^2+\frac{1}{30}c_2(W_i)\right)}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\Bigg\}^{(8)}.
\end{aligned}
\tag{2.22}
$$

Since $\mathrm{ch}(W_i)$ only contribute degree $4l$ forms, we can replace $\cosh\left(\frac{c}{2}\right)$ by $e^{\frac{c}{2}}$, (2.22) gives us (0.7). Q.E.D.

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