Effective vanishing theorems for ample and globally generated vector bundles

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Abstract

By proving an integral formula of the curvature tensor of $E\otimes\det E$, we observe that the curvature of $E\otimes\det E$ is very similar to that of a line bundle and obtain certain new Kodaira-Akizuki-Nakano type vanishing theorems for vector bundles. As special cases, we deduce vanishing theorems for ample, nef and globally generated vector bundles by analytic method instead of the Leray-Borel-Le Potier spectral sequence.

1 Introduction

Many vanishing theorems have been obtained for the Dolbeault cohomology of ample and globally generated vector bundles on smooth projective manifolds, mainly due to the efforts of J. Le Potier, M. Schneider, T. Peternell, A.J. Sommese, J-P. Demailly, L. Ein and R. Lazasfeld, L. Manivel, F. Laytimi and W. Nahm([4], [7], [12], [13],[14],[15], [18], [20], [21]). The Le Potier vanishing theorem says that if E is an ample vector bundle over a smooth projective manifold X, then $H^{p,q}(X,E)=0$ for any $p+q\geq n+r$ where $n=\dim_{\mathbb{C}}X$ and $r=\mathrm{rank}(E)$. When $r\leq n$, the vanishing pairs (p,q) are contained in a **triangle** enclosed by three lines $p+q=n+r$, $p=n$ and $q=n$. By using the Leray-Borel-Le Potier spectral sequence, many interesting generalizations are obtained for products of symmetric and skew-symmetric powers of an ample vector bundle, twisted by a suitable power of its determinant line bundle, see for examples, [4], [18], [12], [13] and [14]. The common feature of their results is that the vanishing theorems hold for (p,q) lying inside or on certain triangles.

As is well-known, except Nakano's vanishing theorem, few vanishing theorems for vector bundles are proved by analytic method. In this paper, we use analytic method to prove vanishing theorems for certain Dolbeault cohomology groups of the bounded vector bundles. The new vanishing theorems have quite different features and they hold for (p,q) lying inside or on certain symmetric **quadrilaterals**.

Definition 1.1. Let E be an arbitrary holomorphic vector bundle with rank r, L an ample line bundle and $\varepsilon_1,\varepsilon_2\in\mathbb{R}$. E is said to be $(\varepsilon_1,\varepsilon_2)$-bounded by L if there exists a Hermitian metric h on E and a *positive* Hermitian metric h^L on L such that the curvature of E is bounded by the curvatures of L^{ε_1} and L^{ε_2}, i.e.

$$\varepsilon_1\omega_L\otimes Id_E\leq\Theta^{E,h}\leq\varepsilon_2\omega_L\otimes Id_E\tag{1.1}$$

in the sense of Griffiths. E is called *strictly* $(\varepsilon_1,\varepsilon_2)$-bounded by L if at least one of $\Theta^{E,h}-\varepsilon_1\omega_L\otimes Id_E$ and $\Theta^{E,h}-\varepsilon_2\omega_L\otimes Id_E$ is not identically zero.

It is easy to see that, if E is $(\varepsilon_1,\varepsilon_2)$-bounded by L, then

$$E\otimes L^{-\varepsilon_1}\quad\text{and}\quad E^*\otimes L^{\varepsilon_2}$$

are semi-positive in the sense of Griffiths. In particular, if $\det E$ is ample, one can choose $L=\det E$ as a natural bound for E. Hence, Definition 1.1 works naturally for many vector bundles in algebraic geometry. We list some examples as follows. See Proposition 3.2 for more details.

(1) If E is globally generated, E is strictly $(0,1)$-bounded by $L\otimes\det E$ for any ample line bundle L;

(2) If E is an ample vector bundle with rank r, then E is strictly $(-1,r)$-bounded by $\det E$;

(3) If E is nef with rank r, then E is strictly $(-1,r)$-bounded by $L\otimes\det E$ for arbitrary ample line bundle L;

(4) If E is Griffiths-positive, E is strictly $(0,1)$-bounded by $\det E$.

Now we describe our main results briefly.

Theorem 1.2. *If E is strictly $(\varepsilon_1,\varepsilon_2)$-bounded by L and $m+(r+k)\varepsilon_1>0$, then*

$$H^{p,q}(X,S^kE\otimes\det E\otimes L^m)=H^{q,p}(X,S^kE\otimes\det E\otimes L^m)=0,\tag{1.2}$$

if $p\geq 1, q\geq 1$ satisfy

$$\min\left\{\frac{n-q}{p},\frac{n-p}{q}\right\}\leq\frac{m+(r+k)\varepsilon_1}{m+(r+k)\varepsilon_2}.\tag{1.3}$$

In particular, $S^kE\otimes\det E\otimes L^m$ is both Nakano-positive and dual-Nakano-positive. Hence

$$H^{n,q}(X,S^kE\otimes\det E\otimes L^m)=H^{q,n}(X,S^kE\otimes\det E\otimes L^m)=0$$

for $q\geq 1$.

Remark 1.3. (1) (p,q) satisfies condition (1.3) if only if it lies inside or on the following quadrilateral $Q=A_0A_1A_2A_3$. See Figure 1 with A_0 and A_2 removed. Here

$$A_0=(0,n),A_1=(n,n),A_2=(n,0),A_3=(c_0,c_0)$$

and

$$c_0=\frac{n}{1+\frac{m+(r+k)\varepsilon_1}{m+(r+k)\varepsilon_2}}.\tag{1.4}$$

It is obvious that Q is symmetric with respect to the line $p=q$.

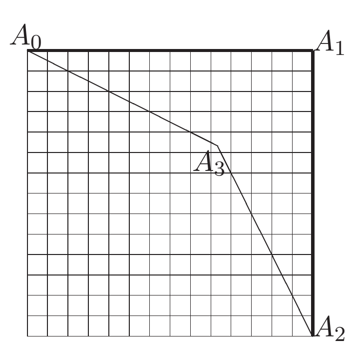

Figure 1

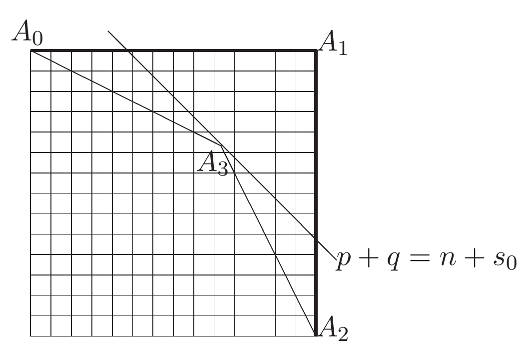

Figure 2

(2) The condition $m+(r+k)\varepsilon_1>0$ is necessary, which guarantees that the vector bundle $S^kE\otimes\det E\otimes L^m$ is Griffiths-positive. In fact, in terms of Hermitian metrics,

$$S^kE\otimes\det E\otimes L^m=S^k(E\otimes L^{-\varepsilon_1})\otimes\det(E\otimes L^{-\varepsilon_1})\otimes L^{m+(r+k)\varepsilon_1}\geq L^{m+(r+k)\varepsilon_1}$$

and similarly $S^kE\otimes\det E\otimes L^m\leq L^{m+(r+k)\varepsilon_2}$. On the other hand, we will see that the bundle $S^kE\otimes\det E\otimes L^m$ has a nice metric h such that $(S^kE\otimes\det E\otimes L^m,h)$ behaves very similarly to a positive Hermitian "line bundle" $(\mathcal{L},h_0)$. Moreover, $m+(r+k)\varepsilon_1$ and $m+(r+k)\varepsilon_2$ are the minimal and maximal eigenvalues of the curvature of $(\mathcal{L},h_0)$ respectively. From these one can see that Theorem 1.2 is optimal.

(3) When ε_1 is very close to ε_2, E is approximate Hermitian-Einstein([10, Chapter IV, §5]) and so it is semi-stable with respect to L ([10, Chapter V, Theorem 8.6]). Moreover, $H^{p,q}(X,S^kE\otimes\det E\otimes L^m)=0$ for any $p+q\geq n+1$.

(4) If $\varepsilon_1\leq 0$, $\varepsilon_2\geq 0$, and F is an arbitrary nef line bundle, Theorem 1.2 also holds for $S^kE\otimes\det E\otimes L^m\otimes F$.

As applications, we obtain

Theorem 1.4. *If E is a globally generated vector bundle with rank r and L is an ample line bundle, then for any $k\geq 1,m\geq 1$,*

$$H^{p,q}(X,S^kE\otimes(\det E)^m\otimes L)=H^{q,p}(X,S^kE\otimes(\det E)^m\otimes L)=0$$

if $p\geq 1,q\geq 1$ satisfy

$$\min\left\{\frac{n-q}{p},\frac{n-p}{q}\right\}\leq\frac{m-1}{m-1+(r+k)}. \tag{1.5}$$

In particular, $S^kE\otimes(\det E)^m\otimes L$ is both Nakan-positive and dual-Nakano-positive and

$$H^{n,q}(X,S^kE\otimes(\det E)^m\otimes L)=H^{q,n}(X,S^kE\otimes(\det E)^m\otimes L)=0$$

for any $q\geq 1$.

The right hand side of (1.5) depends only on the ratios and it makes Theorem 1.4 quite different from the results in [4], [18], [12] and [13]. More precisely, for some specific vanishing pair (p,q), the power of $\det E$ may be independent of the dimension of X. For example, let $n=3n_0+2$, $k=1$ and $m=r+2$. By (1.5), we can choose two different pairs $(p,q)=(2,n-1)$ and $(p,q)=(2n_0+2,2n_0+1)$, and obtain

$$H^{2,n-1}(X,E\otimes(\det E)^{r+2}\otimes L)=0=H^{2n_0+2,2n_0+1}(X,E\otimes(\det E)^{r+2}\otimes L) \tag{1.6}$$

for any globally generated E and ample L. In general, we do not have $H^{p,q}(X,E\otimes(\det E)^{r+2}\otimes L)=0$ **for all** $p+q\geq n+1$, if $1<r\ll n$(cf. [18], Corollary B and [13], Corollary 1.5). On the other hand, for fixed (k,m), the quadrilateral Q contains a triangle $p+q\geq n+s_0$ for some $s_0\in(0,n]$. See Figure 2. Moreover, if the power m of $\det E$ is large enough, we obtain $H^{p,q}(X,S^kE\otimes(\det E)^m\otimes L)=0$ for $p+q\geq n+1$. Examples in [20] and [4] indicate that a sufficient large power of $\det E$ is necessary in this case. For more details, see Corollary 3.8, Corollary 3.10 and Example 4.2.

Theorem 1.5. *Let $r = rank(E)$. If E is ample (resp. nef) and L is nef (resp. ample), then for any $k \geq 1$ and $m \geq k + r + 1$,*

$$H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = H^{q,p}(X, S^k E \otimes (\det E)^m \otimes L) = 0,$$

if $p \geq 1, q \geq 1$ satisfy

$$\min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \leq \frac{(m-1)-(r+k)}{(m-1)+r(r+k)}. \tag{1.7}$$

By a similar setting as (1.6), it is easy to see that the result in Theorem 1.5 is different from the results of [4], [18], [12] and [13].

Remark: Our method is a generalization of the analytic proof of the Kodaira-Akizuki-Nakano vanishing Theorem for line bundles. We have obtained similar results for "partially" positive vector bundles.

2 Background material

Let E be a holomorphic vector bundle over a compact Kähler manifold X and h a Hermitian metric on E. There exists a unique connection ∇ which is compatible with the metric h and complex structure on E. It is called the Chern connection of (E, h). Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R \in \Gamma(X, \Lambda^2 T^*X \otimes E^* \otimes E)$ has the form

$$R = \frac{\sqrt{-1}}{2\pi} R^{\gamma}_{i\bar{j}\alpha} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma \tag{2.1}$$

where $R^{\gamma}_{i\bar{j}\alpha} = h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}$ and

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{2.2}$$

Here and henceforth we adopt the Einstein convention for summation.

Definition 2.1. A Hermitian vector bundle (E, h) is said to be *Griffiths-positive*, if for any nonzero vectors $u = u^i \frac{\partial}{\partial z^i}$ and $v = v^\alpha e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^i \bar{u}^j v^\alpha \bar{v}^\beta > 0 \tag{2.3}$$

(E, h) is said to be *Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0 \tag{2.4}$$

(E, h) is said to be *dual-Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^{i\beta} \bar{u}^{j\alpha} > 0 \tag{2.5}$$

It is easy to see that (E,h) is dual-Nakano-positive if and only if (E^*,h^*) is Nakano-negative. The notions of semi-positivity, negativity and semi-negativity can be defined similarly. We say E is Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, $\cdots$), if it admits a Nakano-positive(resp. Griffiths-positive, dual-Nakano-positive, $\cdots$) metric.

The following analytic definition of nefness is due to [6].

Definition 2.2. Let (X,ω_0) be a compact Kähler manifold. A line bundle L over X is said to be nef, if for any $\varepsilon > 0$, there exists a smooth Hermitian metric h_ε on L such that the curvature of (L,h_ε) satisfies

$$R = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\varepsilon \geq -\varepsilon\omega_0 \tag{2.6}$$

This means that the curvature of L can have an arbitrarily small negative part. Clearly a nef line bundle L satisfies

$$\int_C c_1(L) \geq 0$$

for all irreducible curves $C \subset X$. For projective algebraic S both notions coincide.

By the Kodaira embedding theorem, we have the following analytic definition of ampleness.

Definition 2.3. Let (X,ω_0) be a compact Kähler manifold. A line bundle L over X is said to be ample, if there exists a smooth Hermitian metric h on L such that the curvature R of (L,h) satisfies

$$R = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h > 0 \tag{2.7}$$

Definition 2.4. Let E be a Hermitian vector bundle of rank r over a compact Kähler manifold X, $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ be the tautological line bundle on the projective bundle $\mathbb{P}(E^*)$ and π the canonical projection $\mathbb{P}(E^*) \to X$. By definition([9]), E is an ample vector bundle over X if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is an ample line bundle over $\mathbb{P}(E^*)$. E is said to be nef, if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is nef.

For comprehensive descriptions of positivity, nefness, ampleness and related topics, see [2], [6], [11].

In the following, we will describe the idea of proving vanishing theorems by using an analytic method. Let $(\varphi_{i\overline{j}})_{n\times n}$ be a Hermitian positive matrix with eigenvalues

$$\lambda_1 \leq \cdots \leq \lambda_n \tag{2.8}$$

Let $u = \sum u_{I\overline{J}} dz^I \wedge d\overline{z}^J$ be a (p,q) form on $\mathbb{C}^n$ where $u_{I\overline{J}}$ is alternate in the indices $I = (i_1,\cdots,i_p)$ and $J = (j_1,\cdots,j_q)$. We define

$$T(u,u) = \langle[\varphi,\Lambda_\omega]u,u\rangle \tag{2.9}$$

where $\varphi = \sqrt{-1}\varphi_{i\overline{j}}dz^i \wedge d\overline{z}^j$ and Λ_ω is the contraction operator of the standard Kähler metric on $\mathbb{C}^n$. The following linear algebraic result is obivous([2], p. 334):

Lemma 2.5. *We have the following estimate*

$$T(u,u) \geq \max\{p\lambda_1 - (n-q)\lambda_n, q\lambda_1 - (n-p)\lambda_n\}|u|^2 \tag{2.10}$$

The following result is well-known.

Corollary 2.6. *Let (L, h) be a Hermitian line bundle over a compact Kähler manifold (X, ω_0). Let λ_1 and λ_n be the smallest and largest eigenvalue functions of R^L with respect to ω_0 respectively. If*

$$\max\{p\lambda_1 - (n-q)\lambda_n, q\lambda_1 - (n-p)\lambda_n\}$$

is positive everywhere, then

$$H^{p,q}(M, L) = H^{q,p}(M, L) = 0 \tag{2.11}$$

Proof. By a well-known Bochner formula for L,

$$\Delta'' = \Delta' + [R^L, \Lambda_{\omega_0}]$$

for any $u \in \Omega^{p,q}(M, L)$,

$$\langle \Delta'' u, u\rangle = \langle \Delta' u, u\rangle + T(u,u) \tag{2.12}$$

If $\Delta'' u = 0$, we get $u = 0$ since $T(u,u) \geq 0$. □

Remark 2.7. The condition in Corollary 2.6 can be satisfied if and only if (L, h) is Griffiths positive or Griffiths-negative. If (L, h) is a positive line bundle over a compact complex manifold X, we can define a Kähler metric on X

$$\omega_0 = R^L = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h \tag{2.13}$$

In this case, $\varphi = R^L$ in Lemma 2.6 and $\lambda_1 = \lambda_n = 1$. Hence, if $p+q \geq n+1$, $H^{p,q}(X, L) = 0$. This is the Kodaira-Akizuki-Nakano vanishing theorem. But in general, if R^L is not related to ω_0, we can only get a part of vanishing cohomology groups by this method. More precisely, we can only obtain a vanishing quadrilateral as Figure 1.

Let (E, h) be a Hermitian holomorphic vector bundle with rank r over a compact Kähler manifold (X, ω_g). For any fixed point $p \in X$, there exist local holomorphic coordinates $\{z^i\}_{i=1}^n$ and local holomorphic frames $\{e_\alpha\}_{\alpha=1}^r$ such that

$$g_{i\overline{j}}(p) = \delta_{ij}, \quad h_{\alpha\overline{\beta}}(p) = \delta_{\alpha\overline{\beta}} \tag{2.14}$$

The curvature term in the formula $\Delta'' = \Delta' + [R^E, \Lambda_g]$ can be written as

$$\begin{aligned} T(u,u) &= \langle [R^E, \Lambda_g]u, u\rangle \\ &= \sum R_{i\overline{j}\alpha\overline{\beta}} u_{I,\overline{iS},\alpha}\overline{u}_{I,\overline{jS}\beta} + \sum R_{i\overline{j}\alpha\overline{\beta}} u_{jR,\overline{J},\alpha}\overline{u}_{iR,\overline{J},\beta} - \sum R_{ii\alpha\overline{\beta}} u_{I\overline{J}\alpha}\overline{u}_{I\overline{J}\beta} \end{aligned} \tag{2.15}$$

for any $u = \sum u_{I\overline{J}\alpha} dz^I \wedge d\overline{z}^J \otimes e_\alpha$. For more details, see ([2], p. 341). From formula (2.15), it is very difficult to obtain vanishing theorems for vector bundles. If the curvature R^E has a nice expression, for example

$$R_{i\overline{j}\alpha\overline{\beta}} = \varphi_{i\overline{j}}\tau_\alpha\overline{\tau}_\beta \tag{2.16}$$

then E behaviors as a line bundle with curvature $(\varphi_{i\bar{j}})$. Unfortunately, few examples with property (2.16) can be found(Note also that the curvature formulation here is stronger than the curvature of projectively flat vector bundles). However, an integral version of (2.16) exists on vector bundles of type $E \otimes \det E$,

$$R^{E\otimes \det E}_{i\bar{j}\alpha\bar{\beta}}(s) = R_{i\bar{j}\alpha\bar{\beta}}(s) + \delta_{\alpha\beta}\cdot \sum_{\gamma} R_{i\bar{j}\gamma\bar{\gamma}}(s) = r!\cdot \int_{\mathbb{P}^{r-1}} \frac{\varphi_{i\bar{j}} W_\alpha \overline{W}_\beta}{|W|^2}\frac{\omega_{FS}^{r-1}}{(r-1)!} \tag{2.17}$$

where $[W_1, \cdots, W_r]$ are the homogeneous coordinates on $\mathbb{P}^{r-1}$, ω_{FS} is the Fubini-Study metric and

$$\varphi_{i\bar{j}} = (r+1)\sum_{\gamma,\delta} R_{i\bar{j}\gamma\bar{\delta}}(s)\frac{W_\delta \overline{W}_\gamma}{|W|^2} \tag{2.18}$$

It is obvious that if E is Griffiths-positive, then $E \otimes \det E$ is both Nakano-positive and dual-Nakano-positive. With the help of the nice formulation (2.17), we obtain vanishing theorems similar to Corollary 2.6 for vector bundles.

3 Vanishing theorems for bounded vector bundles

Firstly, we would like to recall the following

Definition 3.1. Let E be an arbitrary holomorphic vector bundle with rank r, L an ample line bundle and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. E is said to be $(\varepsilon_1, \varepsilon_2)$-bounded by L if there exists a Hermitian metric h on E and a *positive* Hermitian metric h^L on L such that the curvature of E is bounded by the curvatures of L^{ε_1} and L^{ε_2}, i.e.

$$\varepsilon_1\omega_L \otimes Id_E \leq \Theta^{E,h} \leq \varepsilon_2\omega_L\otimes Id_E \tag{3.1}$$

in the sense of Griffiths. E is called *strictly* $(\varepsilon_1, \varepsilon_2)$-bounded by L if at least one of $\Theta^{E,h} - \varepsilon_1\omega_L\otimes Id_E$ and $\Theta^{E,h} - \varepsilon_2\omega_L\otimes Id_E$ is not identically zero.

It is easy to see that E is $(\varepsilon_1, \varepsilon_2)$-bounded by L if and only if $E\otimes L^{-\varepsilon_1}$ and $E^*\otimes L^{\varepsilon_2}$ are Griffiths-semi-positive. Similarly, if E is strictly $(\varepsilon_1, \varepsilon_2)$-bounded by L, then at least one of the Griffiths-semi-positive vector bundles $E\otimes L^{-\varepsilon_1}$ and $E^*\otimes L^{\varepsilon_2}$ is not trivial.

Proposition 3.2. *Let E be a holomorphic vector bundle with rank r over a projective manifold.*

(1) If E is globally generated, E is strictly $(0,1)$-bounded by $L\otimes \det E$ for any ample line bundle L;

(2) If E ample, E is strictly $(-1, r)$-bounded by $\det E$;

(3) If E is nef, E is strictly $(-1, r)$-bounded by $L\otimes\det E$ for any ample line bundle L;

(4) If E is Griffiths-positive, E is strictly $(0,1)$-bounded by $\det E$.

Proof. (1) As is well-known, if E is globally generated, there exists a Hermitian metric h on E such that $\Theta^{E,h}$ is Griffiths-semi-positive and $E\otimes\det E^* = \Lambda^{r-1}E^*$ is Griffiths-semi-negative. If L is an ample line bundle, $E\otimes\det E^*\otimes L^*$ is Griffiths-negative, i.e.

$$\Theta^{E,h} < \omega_{L\otimes\det E}\otimes Id_E$$

Hence, E is strictly $(0,1)$-bounded by $L\otimes\det E$.
(2) We assume $r>1$. By a result of [1], [19] and [16], if E is ample, $E\otimes\det E$ is Griffiths-positive. On the other hand, $E^*\otimes\det E = \Lambda^{r-1}E$ is ample and so $(E^*\otimes\det E)\otimes\det(E^*\otimes\det E) = E^*\otimes(\det E)^r$ is Griffiths-positive.
(3) If E is nef, $S^{r+1}E\otimes L$ is ample and by a result of [16], $E\otimes\det E\otimes L$ is Griffiths-positive. Similarly, we know $S^{r+1}(E^*\otimes\det E)\otimes L$ is ample and so $E^*\otimes(\det E)^r\otimes L$ is Griffiths-positive.
(4) It is obvious. $\square$

Remark 3.3. In general, if E is $(-1,r)$-bounded by $\det E$, E is not necessarily ample. For example, let $E = L^3\oplus L^{-1}$ for some ample line bundle L, then E is $(-1,2)$ bounded by $\det E = L^2$.

Let ω_{FS} be the standard Fubini-Study metric on $\mathbb{P}^{r-1}$ with $\int_{\mathbb{P}^{r-1}}\omega_{FS}^{r-1}=1$ and $[W_1,\cdots W_r]$ the homogeneous coordinates on $\mathbb{P}^{r-1}$. If $A=(\alpha_1,\cdots,\alpha_k)$ and $B=(\beta_1,\beta_2,\cdots,\beta_k)$, we define the generalized Kronecker-δ for multi-index by the following formula

$$\delta_{AB} = \sum_{\sigma\in S_k}\prod_{j=1}^{k}\delta_{\alpha_{\sigma(j)}\beta_{\sigma(j)}} \tag{3.2}$$

where S_k is the permutation group in k symbols. The following linear algebraic lemma is obvious (see also [16]).

Lemma 3.4. *If $V_A = W_{\alpha_1}\cdots W_{\alpha_k}$ and $V_B = W_{\beta_1}\cdots W_{\beta_k}$, then*

$$\int_{\mathbb{P}^{r-1}}\frac{V_A\overline{V}_B}{|W|^{2k}}\frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{AB}}{(r+k-1)!} \tag{3.3}$$

For simple-index notations,

$$\int_{\mathbb{P}^{r-1}}\frac{W_\alpha\overline{W}_\beta}{|W|^2}\frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{\alpha\beta}}{r!},\qquad \int_{\mathbb{P}^{r-1}}\frac{W_\alpha\overline{W_\beta}W_\gamma\overline{W_\delta}}{|W|^4}\frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}}{(r+1)!} \tag{3.4}$$

Let h be a Hermitian metric on the vector bundle E. At a fixed point $p\in X$, if we assume $h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$, then the naturally induced bundle $(E\otimes(\det E)^m, h\otimes(\det h)^m)$ has curvature components

$$R^{E\otimes(\det E)^m}_{i\overline{j}\alpha\overline{\beta}} = R_{i\overline{j}\alpha\overline{\beta}} + \delta_{\alpha\beta}\cdot m\sum_\delta R_{i\overline{j}\delta\overline{\delta}} \tag{3.5}$$

where $R_{i\overline{j}\alpha\overline{\beta}}$ is the curvature component of (E,h). It is obvious that S^kE has basis

$$\{e_A = e_1^{\alpha_1}\otimes\cdots\otimes e_r^{\alpha_r}\} \tag{3.6}$$

if $A = (\alpha_1, \cdots, \alpha_r)$ with $\alpha_1 + \cdots + \alpha_r = k$ and α_j are nonnegative integers. Similarly, $(S^k E \otimes (\det E)^m, S^k h \otimes (\det h)^m)$ has curvature components

$$R^{S^k E \otimes (\det E)^m}_{i\bar{j}A\overline{B}} = R_{i\bar{j}A\overline{B}} + \delta_{AB} \cdot m \sum_{\delta} R_{i\bar{j}\delta\bar{\delta}}. \tag{3.7}$$

Lemma 3.5. *If (E, h) is a Hermitian vector bundle, the curvature of $(S^k E \otimes (\det E)^m, S^k h \otimes (\det h)^m)$ can be written as*

$$R^{S^k E \otimes (\det E)^m}_{i\bar{j}A\overline{B}}(p) = (r + k - 1)! \cdot \int_{\mathbb{P}^{r-1}} \frac{V_A \overline{V}_B}{|W|^{2k}} \varphi_{i\bar{j}} \frac{\omega_{FS}^{r-1}}{(r-1)!} \tag{3.8}$$

where

$$\varphi_{i\bar{j}} = (r + k) \sum_{\gamma,\delta} R_{i\bar{j}\gamma\bar{\delta}}(p) \frac{W_\delta \overline{W}_\gamma}{|W|^2} + (m - 1) \sum_{\delta} R_{i\bar{j}\delta\bar{\delta}}. \tag{3.9}$$

Proof. It follows from Lemma 3.4. □

Now we prove Theorem 1.2.

Theorem 3.6. *If E is strictly $(\varepsilon_1, \varepsilon_2)$-bounded by L and $m + (r + k)\varepsilon_1 > 0$, then*

$$H^{p,q}(X, S^k E \otimes \det E \otimes L^m) = H^{q,p}(X, S^k E \otimes \det E \otimes L^m) = 0 \tag{3.10}$$

if $p \geq 1, q \geq 1$ satisfy

$$\min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \leq \frac{m + (r + k)\varepsilon_1}{m + (r + k)\varepsilon_2}. \tag{3.11}$$

In particular, if $m + (r + k)\varepsilon_1 > 0$, $S^k E \otimes \det E \otimes L^m$ is both Nakano-positive and dual-Nakano-positive and

$$H^{n,q}(X, S^k E \otimes \det E \otimes L^m) = H^{q,n}(X, S^k E \otimes \det E \otimes L^m) = 0$$

for $q \geq 1$.

Proof. Let h be a Hermitian metric on E and h^L a positive Hermitian metric on L such that

$$\varepsilon_1 \omega_L \otimes Id_E \leq \Theta^{E,h} \leq \varepsilon_2 \omega_L \otimes Id_E.$$

We can polarize X by

$$\omega_g = \omega_L = -\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log h^L. \tag{3.12}$$

At a fixed point $p \in X$, we can assume

$$g_{i\bar{j}}(p) = \delta_{i\bar{j}} \quad \text{and} \quad h_{\alpha\bar{\beta}}(p) = \delta_{\alpha\bar{\beta}}.$$

Therefore,

$$g_{i\bar{j}}(p) = R^{h^L}_{i\bar{j}}(p) = \delta_{i\bar{j}}. \tag{3.13}$$

Let

$$\varphi_{i\bar{j}} = (r+k)\left(\sum_{\gamma,\delta} R^h_{i\bar{j}\gamma\bar{\delta}}(p)\frac{W_\delta \overline{W}_\gamma}{|W|^2}\right) + mR^{h_L}_{i\bar{j}}, \tag{3.14}$$

then by Lemma 3.5, the curvature tensor of $S^kE\otimes \det E\otimes L^m$ can be written as

$$R^{S^kE\otimes \det E\otimes L^m}_{i\bar{j}A\overline{B}}(p) = (r+k-1)!\cdot \int_{\mathbb{P}^{r-1}} \frac{V_A\overline{V}_B}{|W|^{2k}}\varphi_{i\bar{j}}\frac{\omega_{FS}^{r-1}}{(r-1)!}. \tag{3.15}$$

By formula (3.14), it is easy to see that, if E is $(\varepsilon_1,\varepsilon_2)$-bounded by L, then for any $v = (v^1,\cdots,v^n)\in\mathbb{C}^n\setminus\{0\}$,

$$(m+(r+k)\varepsilon_1)\,|v|^2 \le \varphi_{i\bar{j}}v^i\overline{v}^j \le (m+(r+k)\varepsilon_2)\,|v|^2. \tag{3.16}$$

Since $m+(r+k)\varepsilon_1>0$, it is obvious that $S^kE\otimes\det E\otimes L^m$ is both Nakano-positive and dual-Nakano-positive by (3.15). Let λ_1 be the smallest eigenvalue of $(\varphi_{i\bar{j}})$ and λ_n the largest one, then

$$m+(r+k)\varepsilon_1 \le \lambda_1 \le \lambda_n \le m+(r+k)\varepsilon_2. \tag{3.17}$$

Let $\varphi = \frac{\sqrt{-1}}{2\pi}\varphi_{i\bar{j}}dz^i\wedge d\overline{z}^j$. We consider the curvature term in the Bochner formula

$$\Delta'' = \Delta' + [R,\Lambda_g]$$

for vector bundle $S^kE\otimes\det E\otimes L^m$. For any nonzero

$$u = u_{I\overline{J}A}dz^I\wedge d\overline{z}^J\otimes e_A \in \Omega^{p,q}(X,S^kE\otimes\det E\otimes L^m),$$

we set

$$U=\sum_A u_{I\overline{J}A}V_Adz^I\wedge d\overline{z}^J,$$

then the curvature term can be written as

$$\begin{aligned}
T(u,u) &= \langle [R,\Lambda_g]u,u\rangle\\
&= (r+k-1)!\int_{\mathbb{P}^{r-1}}\langle[\varphi,\Lambda_g]U,U\rangle\cdot\frac{1}{|W|^{2k}}\cdot\frac{\omega_{FS}^{r-1}}{(r-1)!}\\
&\ge (r+k-1)!\int_{\mathbb{P}^{r-1}}\max\{p\lambda_1-(n-q)\lambda_n, q\lambda_1-(n-p)\lambda_n\}|U|^2\cdot\frac{1}{|W|^{2k}}\cdot\frac{\omega_{FS}^{r-1}}{(r-1)!}\\
&= \max\{pK_1-(n-q)K_n, qK_1-(n-p)K_n\}
\end{aligned}$$

where

$$K_i = (r+k-1)!\cdot\int_{\mathbb{P}^{r-1}}\frac{|U|^2}{|W|^{2k}}\lambda_i\frac{\omega_{FS}^{r-1}}{(r-1)!},\qquad i=1,n.$$

By (3.17), if $m+(r+k)\varepsilon_1>0$,

$$\frac{m+(r+k)\varepsilon_1}{m+(r+k)\varepsilon_2} < \frac{K_1}{K_n} \tag{3.18}$$

since E is *strictly* $(\varepsilon_1, \varepsilon_2)$-bounded by L. Note that we use the "strictly $(\varepsilon_1, \varepsilon_2)$-bounded" property to obtain the strict inequality in (3.18). If $p \geq 1, q \geq 1$ satisfy

$$\min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \leq \frac{m+(r+k)\varepsilon_1}{m+(r+k)\varepsilon_2} \tag{3.19}$$

we obtain

$$\min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} < \frac{K_1}{K_n}. \tag{3.20}$$

By standard Bochner formulas(e.g. Corollary 2.6), we deduce that

$$H^{p,q}(X, S^kE \otimes (\det E)^m \otimes L) = H^{p,q}(X, S^kE \otimes (\det E)^m \otimes L) = 0$$

if (p, q) satisfies (3.19). The proof of Theorem 3.6 is complete. □

Proof of Theorem 1.4. When $m = 1$, the conclusion is obvious. In this case, the only vanishing pair is $(p, q) = (n, n)$ and it follows from the fact that $H^{n,n}(X, \mathcal{E}) = 0$ if $\mathcal{E}$ is Griffiths-positive. Now we consider $m \geq 2$. If E is a globally generated vector bundle with rank r and L is an ample line bundle, by Proposition 3.2, E is strictly $(0, 1)$-bounded by $L^{\frac{1}{m-1}} \otimes \det E$. Theorem 1.4 follows from Theorem 1.2 for $(\varepsilon_1, \varepsilon_2) = (0, 1)$ and the relation

$$S^kE \otimes (\det E)^m \otimes L = S^kE \otimes \det E \otimes \left(L^{\frac{1}{m-1}} \otimes \det E\right)^{m-1}.$$

Proof of Theorem 1.5. Similarly, assume $m \geq 2$. If E is ample (resp. nef) and L is nef (resp. ample), by Proposition 3.2, E is strictly $(-1, r)$-bounded by $L^{\frac{1}{m-1}} \otimes \det E$. Theorem 1.5 follows from Theorem 1.2 for $(\varepsilon_1, \varepsilon_2) = (-1, r)$.

Similarly, we have,

Theorem 3.7. *Let (E, h) be a Hermitian vector bundle with semi-Griffiths positive (resp. Griffiths positive) curvature and L is an ample (resp. nef) line bundle L, we have*

$$H^{p,q}(X, S^kE \otimes (\det E)^m \otimes L) = H^{q,p}(X, S^kE \otimes (\det E)^m \otimes L) = 0$$

if $p \geq 1, q \geq 1$ satisfy

$$\min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \leq \frac{m-1}{r+k+m-1}.$$

Proof. It follows from part (4) of Proposition 3.2 and Theorem 1.2. □

Now we want to analyze the condition

$$\min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} \leq \lambda_0 \tag{3.21}$$

for some $\lambda_0 \in [0, 1]$. Without loss of generality, we assume $p \geq q \geq 1$, then (3.21) is equivalent to

$$p + \lambda_0 q \geq n. \tag{3.22}$$

When $p = q$, we obtain

$$c_0 = \frac{n}{1+\lambda_0}. \tag{3.23}$$

(p, q) satisfies (3.21) if and only if (p, q) lies in the quadrilateral $Q = A_0A_1A_2A_3$ where

$$A_0 = (0, n), A_1 = (n, n), A_2 = (n, 0), A_3 = (c_0, c_0) \tag{3.24}$$

In the following, we consider the vanishing triangle shown in Figure 2.

Corollary 3.8. *Let E be globally generated and L be ample.*

(1) If the pair (k, m, s) satisfies

$$m \geq \frac{1}{s}\left[\frac{n-s}{2}\right](r+k)+1 \tag{3.25}$$

where $[\bullet]$ is the integer part of $\bullet$, then

$$H^{p,q}(X, S^kE \otimes (\det E)^m \otimes L) = 0$$

for any $p + q \geq n + s$.

(2) For fixed (k, m), we have

$$H^{p,q}(X, S^kE \otimes (\det E)^m \otimes L) = 0$$

for any (p, q) satisfies

$$p+q \geq n + \left(\frac{2n}{1+\frac{m-1}{r+k+m-1}} - n\right). \tag{3.26}$$

Proof. If $m \geq \frac{1}{s}\left[\frac{n-s}{2}\right](r+k)+1$, we get

$$\frac{m-1}{r+k+m-1} \geq \frac{\left[\frac{n-s}{2}\right]}{\left[\frac{n-s}{2}\right]+s}. \tag{3.27}$$

If $p + q \geq n + s$,

$$\max_{p+q\geq n+s} \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\} = \frac{\left[\frac{n-s}{2}\right]}{\left[\frac{n-s}{2}\right]+s}. \tag{3.28}$$

Part (1) follows from Theorem 1.4. For part (2), if

$$p+q \geq n + \left(\frac{2n}{1+\frac{m-1}{r+k+m-1}} - n\right) = \frac{2n}{1+\frac{m-1}{r+k+m-1}}$$

then

$$\max\{p, q\} \geq \frac{n}{1+\frac{m-1}{r+k+m-1}}. \tag{3.29}$$

That is

$$\frac{m-1}{r+k+m-1} \geq \min\left\{\frac{n-q}{p}, \frac{n-p}{q}\right\}.$$

Hence part (2) follows. $\square$

Remark 3.9. Theorem 1.4 and Corollary 3.8 are also valid for semi-Griffiths positive E. Consider the example $E = T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with the canonical metric. Since $r = n = 2$, by Corollary 3.8, we obtain

$$H^{p,q}(X, E \otimes (\det E)^m \otimes L) = 0 \tag{3.30}$$

for any $p + q \geq n + 1$ if $m \geq 1$. It is obvious that the lower bound 1 is sharp since

$$H^{n,n-1}(X, E \otimes L) \cong H^{1,1}(\mathbb{P}^n, \mathbb{C}) = \mathbb{C} \tag{3.31}$$

if we choose $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $m = 1$. So the lower bound

$$\frac{1}{s}\left[\frac{n-s}{2}\right](r+k) + 1$$

can not be improved by a universal constant, i.e., a constant independent on r, s, n, k. Hence the lower bound is optimal in that sense.

Similarly, we obtain

Corollary 3.10. *Let E be ample (resp.) and L be nef (resp. ample). Suppose $k \geq 1$ and $m \geq r + k + 1$.*

(1) If the pair (k, m, s) satisfies

$$m \geq \frac{1}{s}\left[\frac{n-s}{2}\right](r+k)(r+1) + (r+1) + k,$$

then

$$H^{p,q}(X, S^k E \otimes (\det E)^m) = 0$$

for any $p + q \geq n + s$.

(2) For fixed (k, m), we have

$$H^{p,q}(X, S^k E \otimes (\det E)^m \otimes L) = 0$$

for any (p, q) satisfies

$$p + q \geq n + \left(\frac{2n}{1 + \frac{(m-1)-(r+k)}{(m-1)+r(r+k)}} - n\right).$$

4 Examples

It is well-known that globally generated vector bundles are Griffiths semi-positive. On the other hand, any globally generated vector bundle has a quotient metric induced from the trivial vector bundle and so it is semi-dual-Nakano-positive([2]).

Corollary 4.1. *Let E be a globally generated vector bundle and L an ample line bundle over a projective manifold X, then $S^k E \otimes L$ is dual-Nakano-positive for any $k \geq 1$. Moreover,*

$$H^{p,n}(X, S^k E \otimes L) = 0 \tag{4.1}$$

for any $p \geq 1$.

However, in general, we can not obtain a vanishing quadrilateral for $S^kE\otimes L$ as Figure 1. It is easy to see that the result in Corollary 4.1 is a vertical line on the boundary of the quadrilateral in Figure 1. In [20], the authors found more vanishing elements close to that vertical line. More precisely, they proved that

$$H^{p,n-1}(X,S^kE\otimes L)=0, \qquad \text{for any} \quad p\geq r+1 \tag{4.2}$$

But in general, they proved that there exists some $1\leq q\leq n$ such that $H^{n,q}(X,S^kE\otimes L)\neq 0$. In particular, $S^kE\otimes L$ is not necessarily Nakano-positive. For example, $E=T\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(-1)$ and $L=\mathcal{O}_{\mathbb{P}^n}(1)$. It is obvious E is globally generated. When $n\geq 2$, $E\otimes L=T\mathbb{P}^n$ is dual-Nakano-positive but not Nakano-positive. More generally, we have

Example 4.2 (Demailly, [4])**.** Let $X=G(r,V)$ be the Grassmannian of subspaces of codimension r of a vector space V, $\dim_{\mathbb{C}}V=d$, and E the tautological quotient vector bundle of rank r over X. Then E is globally generated and $L:=\det E$ is very ample.

$$H^{n,q}(X,S^kE\otimes\det E)=\begin{cases}0, & q\neq(r-1)(d-r);\\ S^{k+r-d}V\otimes\det V, & q=(r-1)(d-r)\end{cases} \tag{4.3}$$

where $n=\dim_{\mathbb{C}}X=r(d-r)$. If $r=d-1$, then $X=\mathbb{P}^n=\mathbb{P}^{d-1}$ and $E=T\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(-1)$, $\det E=\mathcal{O}_{\mathbb{P}^n}(1)$. That is

$$H^{n,q}(\mathbb{P}^n,S^kT\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(1-k))=\begin{cases}0, & q\neq n-1;\\ S^{k-1}V\otimes\det V, & q=n-1\end{cases} \tag{4.4}$$

Therefore, if $n\geq 2$, $S^kT\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(1-k)$ can not be Nakano-positive by the non-vanishing. However, we shall see that for any $\ell\geq 2-k$, $S^kT\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(\ell)$ is both Nakano-positive and dual-Nakano-positive. Moreover, we can obtain more vanishing results about it.

Let h_{FS} be the Fubini-Study metric on $\mathbb{P}^n$ and it also induces a metric on $L=\mathcal{O}_{\mathbb{P}^n}(1)$. It is easy to see that

$$\omega_L\otimes Id\leq\Theta^{T\mathbb{P}^n}\leq 2\omega_L\otimes Id. \tag{4.5}$$

So $T\mathbb{P}^n$ is strictly $(1,2)$-bounded by L. Similarly, $H=T\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(-1)$ is strictly $(0,1)$-bounded by L.

Proposition 4.3. *If $\ell\geq 2-k$, $S^kT\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(\ell)$ is Nakano-positive and dual-Nakano-positive and*

$$H^{p,q}(\mathbb{P}^n,S^kT\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(\ell))=H^{q,p}(\mathbb{P}^n,S^kT\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(\ell))=0 \tag{4.6}$$

for any $p\geq 1, q\geq 1$ satisfy

$$\min\left\{\frac{n-p}{q},\frac{n-q}{p}\right\}\leq\frac{\ell+k-1}{\ell+n+2k-1}. \tag{4.7}$$

Proof. Since $\det(T\mathbb{P}^n)=\mathcal{O}_{\mathbb{P}^n}(n+1)$, we see $\det(H)=\mathcal{O}_{\mathbb{P}^n}(1)$. It follows from the relation

$$S^kH\otimes\det H\otimes\mathcal{O}_{\mathbb{P}^n}(\ell+k-1)=S^kT\mathbb{P}^n\otimes\mathcal{O}_{\mathbb{P}^n}(\ell) \tag{4.8}$$

and Theorem 1.2 with $(\varepsilon_1,\varepsilon_2)=(0,1)$ and $m=r+k-1$. Here $\ell+k-1\geq 1$, i.e., $\ell\geq 2-k$ is necessary and optimal by Example 4.2. ∎

Remark 4.4. Although $T\mathbb{P}^n$ is not Nakano-positive when $n \geq 2$, $S^k T\mathbb{P}^n$ is both Nakano-positive and dual-Nakano-positive for any $k \geq 2$ (see [16]). It is also easy to see that similar results as Proposition 4.3 hold on general flag manifolds.

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