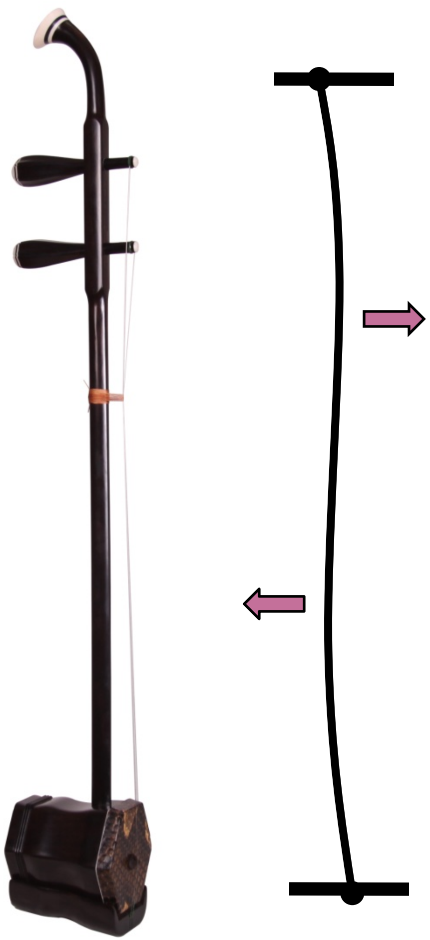
第二讲:
弦振动方程的导出

1. 模型过程



(1) 问题：两端固定的张紧的细弦，作微小的横振动，求其振动规律

(2) 假设：

A 弦均匀且细（视为曲线）；

B 作微小的横振动；

C 柔软（忽略变形应力）

(3) 用到的基本原理：

牛顿第二定律： $f = ma = m\frac{dv}{dt}$

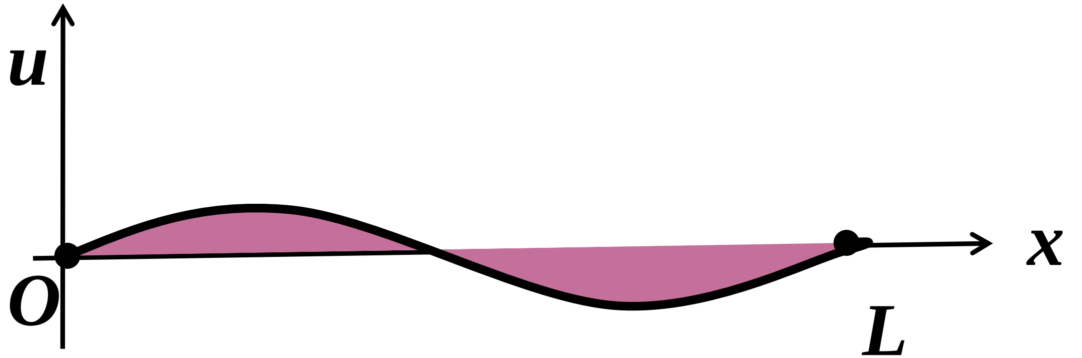
$$f \cdot \Delta t = m\frac{\Delta v}{\Delta t} \cdot \Delta t = m\Delta v = \Delta(mv)$$

（冲量=动量的变化）

(4) 数学描述（建立坐标系）：

$u(x,t)$—位移

$T(x,t)$ —张力



（5）模型构建：

首要目标：$T(x,t) \approx T^*(x) \approx T_0$

在弦上取小段$(x, x+\Delta x)$，则其弧长为：

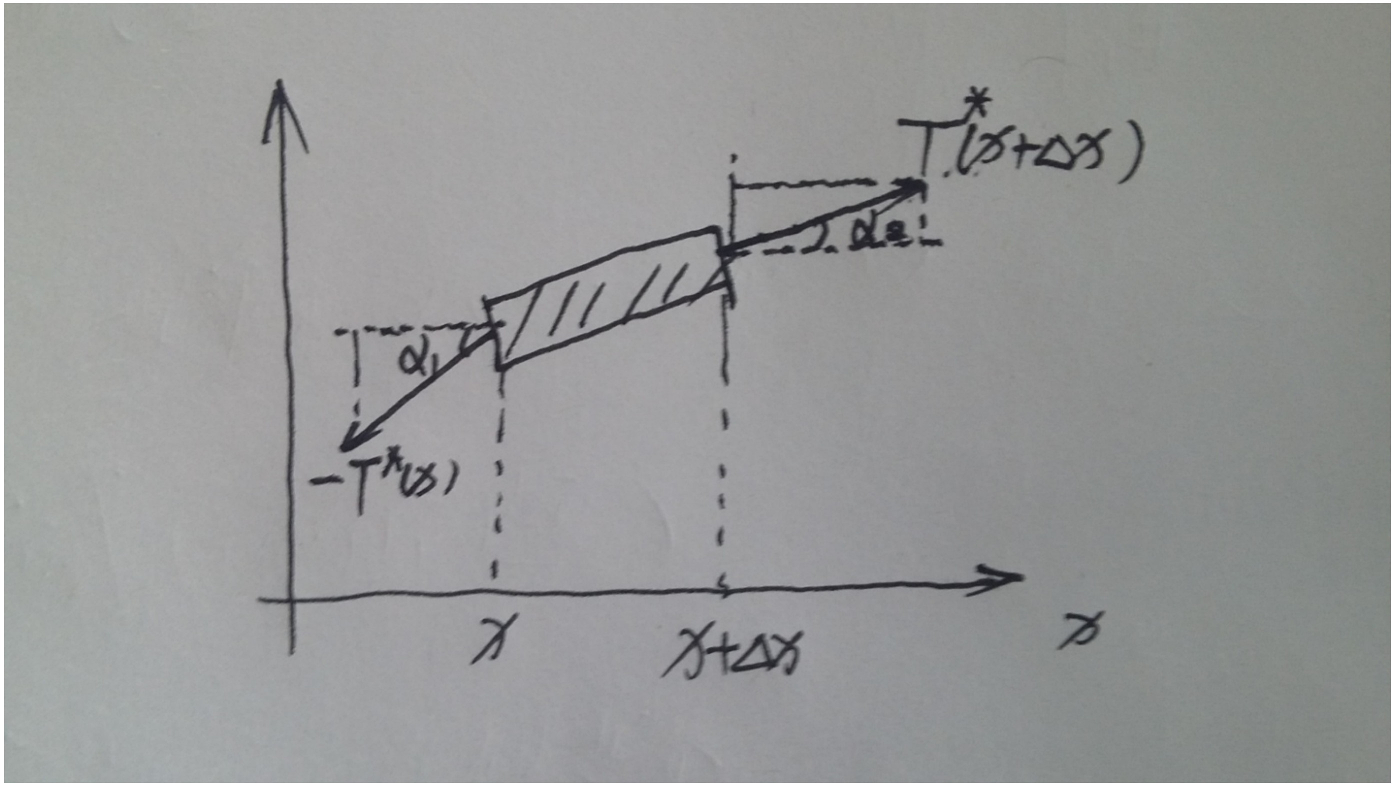
$$\Delta s = \int_x^{x+\Delta x} \sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\, dx \approx \int_x^{x+\Delta x} 1 dx = \Delta x$$

[微小横振动假设，斜率小]

从而弦在振动过程中几乎没有伸长

【胡克定律$F = -k\,s$】

$\therefore \quad T(x,t) \approx T^*(x)$



作受力分析：

水平向：$F_x = T*(x+\Delta x)\cos\alpha_2 - T*(x)\cos\alpha_1 \approx 0$

[微小横振动假设]

垂向：

$$F_u = T^*(x+\Delta x)\sin\alpha_2 - T^*(x)\sin\alpha_1$$

$$\approx T_0[\sin\alpha_2 - \sin\alpha_1]$$

$$\approx T_0\left[\frac{\sin\alpha_2}{\cos\alpha_2} - \frac{\sin\alpha_1}{\cos\alpha_1}\right]$$

$$\approx T_0\left[\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - \frac{\partial u}{\partial x}\bigg|_{x}\right]$$

1 **Δt时段内长度为Δx的弦于垂向上合力的总冲量：**

$$\int_t^{t+\Delta t} F_u dt = \int_t^{t+\Delta t} T_0\left[\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - \frac{\partial u}{\partial x}\bigg|_{x}\right]dt$$

$$= \int_t^{t+\Delta t}\int_x^{x+\Delta x} T_0 \frac{\partial^2 u}{\partial x^2} dxdt$$

2 t时刻到$t+\Delta t$时刻长度为Δx的弦动量的增加量：

$$\int_{x}^{x+\Delta x}\left[\rho dx\frac{\partial u}{\partial t}\bigg|_{t+\Delta t}-\rho dx\frac{\partial u}{\partial t}\bigg|_{t}\right]$$

$$=\int_{x}^{x+\Delta x}\rho\left[\frac{\partial u}{\partial t}\bigg|_{t+\Delta t}-\frac{\partial u}{\partial t}\bigg|_{t}\right]dx$$

$$=\int_{x}^{x+\Delta x}\int_{t}^{t+\Delta t}\rho\frac{\partial^2 u}{\partial t^2}\,dtdx$$

$$\int_{x}^{x+\Delta x}\int_{t}^{t+\Delta t}\left[\rho\frac{\partial^2 u}{\partial t^2}-T_0\frac{\partial^2 u}{\partial x^2}\right]dtdx=0$$

由Δx与Δt的任意性知被积函数：

$$\rho\frac{\partial^2 u}{\partial t^2}-T_0\frac{\partial^2 u}{\partial x^2}=0$$

当增加外力$F(x,t)$，例如重力$F=-\rho g$

$$\rho\frac{\partial^2 u}{\partial t^2}-T_0\frac{\partial^2 u}{\partial x^2}=F(x,t)$$

记$a^2=T_0/\rho$，$f(x,t)=F(x,t)/\rho$化为

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0 \quad \text{（齐次方程）}$$

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=f(x,t) \quad \text{（非齐次方程）}$$

二维： $$\frac{\partial^2 u}{\partial t^2} - a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f(x,y,t)$$

（鼓膜振动、水波）

三维： $$\frac{\partial^2 u}{\partial t^2} - a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=f(x,y,z,t)$$

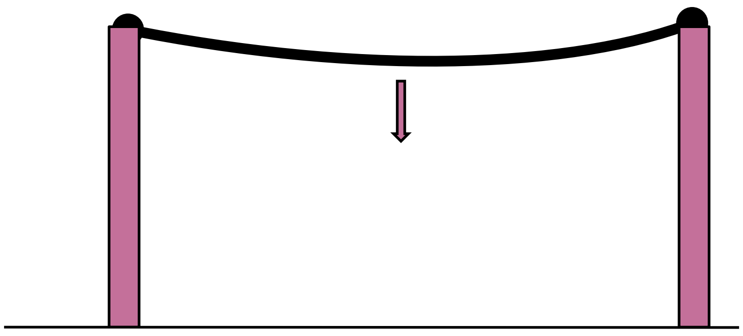
（物体振动、声波）

应用实例：沿公路架设电线，路长S，线密度ρ，工程要求的张力T，电线杆间距L，问线长对这些参数的依赖关系如何？

$$\underbrace{\rho\frac{\partial^2 u}{\partial t^2}}_{\text{忽略}}-T\frac{\partial^2 u}{\partial x^2}=-\rho g$$

$$\Rightarrow \quad -T\frac{\partial^2 u}{\partial x^2}=-\rho g$$

$$\Rightarrow \quad u=\frac{\rho g}{2T}x(x-L)$$



$$S^*=\frac{S}{L}\int_0^L\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}dx=\frac{S}{L}\int_0^L\sqrt{1+\left[\frac{\rho g}{2T}(2x-L)\right]^2}dx$$

谢谢！