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Calabi-Yau Objects in the Stable Category of a Finite Dimensional Pointed Hopf Algebra of Rank One

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Abstract: Let H be an arbitrary finite dimensional pointed Hopf algebra of rank one. We use the results due to Cibils and Zhang to determine the minimal, consequently all Calabi-Yau objects in the stable module category H-$\underline{\text{mod}}$ of H.

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0 Introduction

The notion of Calabi-Yau triangulated category was introduced by Kontsevich[13] in the late 1990's. It appears in mathematical physics[8], non-commutative algebraic geometry[3], and representation theory of quivers and finite-dimensional algebras[4, 19]. Many important triangulated categories admit the Calabi-Yau property, e.g. Calabi-Yau categories arising as orbit categories[12], Calabi-Yau categories arising as stable module categories[2], and Calabi-Yau categories arising as derived categories[10, 18].

It is well known that Calabi-Yau categories have some good properties, e.g. global naturality. However, even in non-Calabi-Yau categories, one can still introduce the notion of a Calabi-Yau object which enjoys similar properties. For instance, Cibils and Zhang in [7, Section 3.1] introduced the concept of a Calabi-Yau object in a Hom-finite Krull-Schmidt triangulated K-category and studied some properties of such objects. In particular, they classified all the d-th Calabi-Yau objects in the stable categories of selfinjective Nakayama algebras for any integer d, and determined all selfinjective Nakayama algebras whose stable categories have indecomposable Calabi-Yau objects.

In this paper, we apply these results appeared in [7] to a particular selfinjective Nakayama algebra, namely, a finite dimensional pointed Hopf algebra H of rank one. We characterize whether or not the stable module category H-$\underline{\text{mod}}$ is Calabi-Yau. Moreover, if H-$\underline{\text{mod}}$ is not Calabi-Yau, we study the Heller's syzygy functor Ω as well as the Nakayama functor $\mathcal{N}$ explicitly, and use them to determine the minimal, consequently all Calabi-Yau objects of H-$\underline{\text{mod}}$ completely.

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1 Finite Dimensional Pointed Hopf Algebras of Rank One

In this section, we recall the construction and classification of any finite dimensional pointed Hopf algebra of rank one in terms of a group datum.

A Hopf algebra H over a field K is called pointed if all its simple left or right comodules are one dimensional. This is equivalent to saying that the coradical of H is a group algebra[16]. Let H_0 be the coradical of Hopf algebra H. We define $H_i := \Delta^{-1}(H \otimes H_{i-1} + H_0 \otimes H)$ for $i \geq 1$. Then $\{H_i \mid i \geq 0\}$ is called the coradical filtration of Hopf algebra H. If H is pointed, then its coradical filtration is a Hopf algebra filtration (see [16, Lemma 5.2.8]).

Let $\{H_i \mid i \geq 0\}$ be the coradical filtration of Hopf algebra H. We assume that the coradical H_0 is a Hopf subalgebra of H. Then each H_i is a free H_0-module. Consider K as the trivial right H_0-module, if H is generated as an algebra by H_1 and $\dim_K(K \otimes_{H_0} H_1) = n+1$, then H is called a Hopf algebra of rank n (see [14]).

A quadruple $\mathcal{D} = (G, \chi, g, \mu)$ is called a group datum if G is a finite group, χ a K-linear character of G, g an element in the center of G, and $\mu \in K$ subject to $\chi^n = 1$ or $\mu(g^n - 1) = 0$, where n is the order of $\chi(g)$. If $\mu(g^n - 1) = 0$, then the group datum $\mathcal{D}$ is said to be of nilpotent type. If $\mu(g^n - 1) \neq 0$ and $\chi^n = 1$, then it is of non-nilpotent type (see [14]).

For any group datum $\mathcal{D} = (G, \chi, g, \mu)$, we denote by $H_{\mathcal{D}}$ the associative algebra generated by y and all h in G such that KG is a subalgebra of $H_{\mathcal{D}}$ and

$$y^n = \mu(g^n - 1), \quad yh = \chi(h)hy$$

for any $h \in G$. In addition, $H_{\mathcal{D}}$ is endowed with a Hopf algebra structure, where the comultiplication $\triangle$, the counit ε, and the antipode S are given respectively by

$$\triangle(y) = y \otimes g + 1 \otimes y, \quad \varepsilon(y) = 0, \quad S(y) = -yg^{-1},$$
$$\triangle(h) = h \otimes h, \quad \varepsilon(h) = 1, \quad S(h) = h^{-1}$$

for all $h \in G$.

The Hopf algebra $H_{\mathcal{D}}$ is of finite dimension with a canonical K-basis $\{y^i h \mid h \in G,\ 0 \leq i \leq n-1\}$. Thus, $\dim H_{\mathcal{D}} = n|G|$, where $|G|$ is the order of G. It is easy to see that G is the group of group-like elements of $H_{\mathcal{D}}$ and $H_{\mathcal{D}}$ is a finite dimensional pointed Hopf algebra of rank one.

Example 1.1 Let G be a cyclic group of order m with a generator $g, \omega \in K$ a primitive m-th root of unity and χ a K-linear character of G given by $\chi(g) = \omega$.

(1) The group datum $\mathcal{D} = (G, \chi, g, \mu)$ is of nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is nothing but a Taft algebra[6].

(2) Suppose that $d > 1$ is a divisor of m. Then the group datum $\mathcal{D} = (G, \chi, g^d, \mu)$ is of nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is a generalized Taft algebra[15].

(3) Suppose that $d > 1$ is a divisor of m. Then the group datum $\mathcal{D} = (G, \chi^d, g, \mu)$ $(\mu \neq 0)$ is of non-nilpotent type and the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is a Radford Hopf algebra[17].

Lemma 1.1 Let $E = \frac{1}{|G|}\sum_{h \in G} h$. Then the spaces of left and right integrals of $H_{\mathcal{D}}$ are spanned respectively by Ey^{n-1} and $y^{n-1}E$.

Proof If the group datum $\mathcal{D}$ is of nilpotent type, then $hEy^{n-1} = Ey^{n-1} = \varepsilon(h)Ey^{n-1}$ for any $h \in G$, and $yEy^{n-1} = 0 = \varepsilon(y)Ey^{n-1}$. Thus, Ey^{n-1} is a non-zero left integral of $H_{\mathcal{D}}$. It is similar that $y^{n-1}E$ is a non-zero right integral of $H_{\mathcal{D}}$. If the group datum $\mathcal{D}$ is of non-nilpotent type, then $hEy^{n-1} = Ey^{n-1} = \varepsilon(h)Ey^{n-1}$ for any $h \in G$. It follows from $\chi(g^n) = 1$ that

$$\begin{aligned} yEy^{n-1} &= \frac{1}{|G|}\sum_{h\in G} yhy^{n-1} \\ &= \frac{1}{|G|}\sum_{h\in G} \chi(h)hy^n \\ &= \frac{1}{|G|}\sum_{h\in G} \chi(h)h\mu(g^n-1) \\ &= \frac{\mu}{|G|}\sum_{h\in G} (\chi(hg^n)hg^n - \chi(h)h) \\ &= 0. \end{aligned}$$

Hence, $yEy^{n-1} = \varepsilon(y)Ey^{n-1}$ and Ey^{n-1} is a left integral of $H_{\mathcal{D}}$. Similarly, $y^{n-1}E$ is a right integral of $H_{\mathcal{D}}$. □

It follows from Lemma 1.2 that the space of left integrals is not equal to the right one, and hence $H_{\mathcal{D}}$ is neither unimodular nor symmetric.

Remark 1.1 If the order of $\chi(g)$ is $n = 1$, then $H_{\mathcal{D}}$ is nothing but the group algebra KG. To avoid this, we always assume that $n \geq 2$ throughout this paper. In this situation $\chi(g) \neq 1$, this implies that $g \neq 1$ and $\chi \neq \varepsilon$.

The family of finite dimensional pointed Hopf algebras of rank one coincides with the family of non-semisimple monomial Hopf algebras appeared in [5]. The classification of such Hopf algebras over an algebraically closed field K of characteristic 0 has been given respectively in [5, 14]. We follow the approach of Krop and Radford and present the classification result as follows (see [14, Theorem 1]):

Proposition 1.1 We have the following classification result:

(1) For any group datum $\mathcal{D}$, the Hopf algebra $H_{\mathcal{D}}$ associated to $\mathcal{D}$ is a finite dimensional pointed Hopf algebra of rank one.

(2) Every finite dimensional pointed Hopf algebra of rank one over an algebraically closed field K of characteristic 0 is isomorphic to $H_{\mathcal{D}}$ for some group datum $\mathcal{D}$.

(3) Let $\mathcal{D} = (G, \chi, g, \mu)$ and $\mathcal{D}' = (G', \chi', g', \mu')$ be two group data. Then $H_{\mathcal{D}}$ and $H_{\mathcal{D}'}$ are isomorphic as Hopf algebras if and only if there is a group isomorphism $f : G \to G'$ such that $f(g) = g'$, $\chi = \chi' \circ f$ and $\beta\mu'(g'^n - 1) = \mu(g'^n - 1)$ for some non-zero $\beta \in K$, where n is the order of $\chi(g)$.

2 Calabi-Yau Objects

In this section, H is always a finite dimensional pointed Hopf algebra of rank one associated to the group datum $\mathcal{D} = (G, \chi, g, \mu)$ over an algebraically closed field K of characteristic 0. We shall use the results of Cibils and Zhang in [7] to determine all Calabi-Yau objects in the stable category H-$\underline{\text{mod}}$ of H.

In the following, we shall list all finite dimensional indecomposable H-modules in both cases the group datum $\mathcal{D}$ is of nilpotent type and of non-nilpotent type. Accordingly, we obtain all indecomposable objects in the stable category H-$\underline{\text{mod}}$ of H.

Let $\{V_i \mid i \in \Omega\}$ be a complete set of non-isomorphic simple KG-modules. Since g is a central element of G, the action of g^n on each V_i is a scalar multiple by a non-zero element, say λ_i. We denote by $\Omega_0 = \{i \in \Omega \mid \lambda_i = 1\}$ and $\Omega_1 = \{i \in \Omega \mid \lambda_i \neq 1\}$. Note that $0 \in \Omega_0$ as we denote by V_0 the trivial H-module K. If the group datum $\mathcal{D}$ is of nilpotent type, then $\Omega = \Omega_0$ by the observation of the definition of a group datum.

Let x be a variable. For any $k \in \mathbb{N}$ and $i \in \Omega$, we define $x^k u + x^k v = x^k(u+v)$ and $\lambda(x^k u) = x^k(\lambda u)$, for any $u, v \in V_i$ and $\lambda \in K$. Then $x^k V_i$ becomes a K-space. For any $i \in \Omega_0$ and $1 \leq j \leq n$, let $M(i,j) := V_i \oplus xV_i \oplus \cdots \oplus x^{j-1}V_i$. Then $M(i,j)$ is an H-module given by

$$h(x^k v) = \chi^{-k}(h) x^k hv, \quad 0 \leq k \leq j-1$$

and

$$y(x^k v) = \begin{cases} x^{k+1}v, & 0 \leq k \leq j-2, \\ 0, & k = j-1 \end{cases}$$

for all $h \in G$ and $v \in V_i$.

For any $j \in \Omega_1$, let $P_j := V_j \oplus xV_j \oplus \cdots \oplus x^{n-1}V_j$. Then P_j is an H-module with the actions given by

$$h(x^k v) = \chi^{-k}(h) x^k hv, \quad 0 \leq k \leq n-1,$$

and

$$y(x^k v) = \begin{cases} x^{k+1}v, & 0 \leq k \leq n-2, \\ (g^n - 1)v, & k = n-1 \end{cases}$$

for all $h \in G$ and $v \in V_j$.

The indecomposable H-modules are completely described as follows (see [20, Theorem 2.5 (4)] and [21, Theorem 2.9] respectively):

Theorem 2.1 We have the following result:

(1) If the group datum $\mathcal{D}$ is of nilpotent type, then the set $\{M(i,j) \mid i \in \Omega_0,\ 1 \leq j \leq n\}$ forms a complete set of finite dimensional indecomposable H-modules up to isomorphism.

(2) If the group datum $\mathcal{D}$ is of non-nilpotent type, then the set $\{M(i,k), P_{[j]} \mid i \in \Omega_0, 1 \leq k \leq n, [j] \in \overline{\Omega}_1\}$ forms a complete set of finite dimensional indecomposable H-modules up to isomorphism, where $\overline{\Omega}_1$ is a set of equivalence classes with respect to an equivalence relation defined on Ω_1.

Note that all indecomposable projective H-modules are $M(i,n)$ for $i \in \Omega_0$ if the group datum $\mathcal{D}$ is of nilpotent type, and $M(i,n)$ for $i \in \Omega_0$ as well as $P_{[j]}$ for $[j] \in \overline{\Omega}_1$ if the group datum $\mathcal{D}$ is of non-nilpotent type (see [20–21]). Now we obtain all indecomposable objects in the stable category H-$\underline{\text{mod}}$ stated as follows:

Corollary 2.1 No matter what the type of the group datum $\mathcal{D}$ is, the set $\{M(i,j) \mid i \in \Omega_0,\ 1 \leq j \leq n-1\}$ forms a complete set of indecomposable objects in the stable category H-$\underline{\text{mod}}$ up to isomorphism.

Since H is a selfinjective algebra, the Nakayama functor $\mathcal{N} := H^* \otimes_H -$, the Heller's syzygy functor Ω, and the Auslander-Reiten translate $\mathrm{DTr} \cong \Omega^2 \circ \mathcal{N} \cong \mathcal{N} \circ \Omega^2$ are endo-equivalences of H-$\underline{\text{mod}}$ (see [1, Chapter IV]). Note that H-$\underline{\text{mod}}$ is a Hom-finite Krull-Schmidt triangulated K-category with the shift functor $[1] = \Omega^{-1}$ (see [11, p. 16]). It follows that the Serre functor of H-$\underline{\text{mod}}$ is $[1] \circ \mathrm{DTr} \cong \Omega \circ \mathcal{N}$ (see [7, Section 2.4]).

Note that H is Frobenius. There are $\lambda \in \int_{H^*}^l$ and $\Lambda \in \int_H^r$ such that $\lambda(\Lambda) = 1$ and λ is a Frobenius homomorphism with the dual basis $\{S(\Lambda_1), \Lambda_2\}$. Since $S(\Lambda) \in \int_H^l$, there is a group-like element α of H^* such that $S(\Lambda)b = \alpha(b)S(\Lambda)$ for any $b \in H$. By Lemma 1.1, we have $Ey^{n-1} \in \int_H^l$, and $Ey^{n-1}h = \chi^{n-1}(h)Ey^{n-1}$, $Ey^{n-1}y = 0$. It follows that

$$\alpha(h) = \chi^{n-1}(h) \ \text{ for } h \in G, \quad \text{and} \quad \alpha(y) = 0.$$

For the non-degenerate associative bilinear form $\langle b, c\rangle := \lambda(bc)$, there is a Nakayama automorphism $\mu : H \to H$ such that $\langle b, c\rangle = \langle \mu(c), b\rangle$ for $b, c \in H$. For any H-module M, we denote by $M^{(\mu)}$ the H-module with underlying K-space M and action $b \cdot u := \mu(b)u$ for $b \in H$ and $u \in M$. Since twisting M by an inner automorphism reproduces M, the Nakayama automorphism induces a naturally equivalent automorphism on the category of finite dimensional left H-modules.

Lemma 2.1 With the notions above, the Nakayama automorphism μ of H is given by $\mu(h) = \chi^{n-1}(h)h$, $\mu(y) = y$ for any $h \in G$.

Proof Using the dual basis $\{S(\Lambda_1), \Lambda_2\}$ with respect to the Frobenius homomorphism λ, we have $\mu(b) = \sum \lambda(\mu(b)S(\Lambda_1))\Lambda_2 = \sum \lambda(S(\Lambda_1)b)\Lambda_2$. Applying S^2 to this equality, and recalling that λ is a left integral in H^*, we obtain that

$$\begin{aligned} S^2(\mu(b)) &= \sum S^2(\Lambda_2)\lambda(S(\Lambda_1)b) \\ &= \sum S^2(\Lambda_3)S(\Lambda_2)b_1\lambda(S(\Lambda_1)b_2) \\ &= \sum b_1\lambda(S(\Lambda)b_2) \\ &= \sum b_1\lambda(\alpha(b_2)S(\Lambda)) \\ &= \sum b_1\alpha(b_2). \end{aligned}$$

Then $\mu(b) = \sum \overline{S}^2(b_1)\alpha(b_2)$, where $b \in H$ and $\overline{S}$ is the inverse of the antipode S under composition. Since $\overline{S}^2(h) = h$, $\overline{S}^2(y) = qy$ and $\alpha(h) = \chi^{n-1}(h)$, $\alpha(y) = 0$, we have $\mu(h) = \chi^{n-1}(h)h$, $\mu(y) = y$, as desired. □

To describe the Calabi-Yau objects in H-mod, we need to determine the Nakayama functor $\mathcal{N}$ and the Heller's syzygy functor Ω of H-modules explicitly. Let $V_{\chi^{-1}}$ denote the simple KG-module corresponding to the K-linear character χ^{-1}. For any simple KG-module V_i with $i \in \Omega_0$, the tensor product $V_{\tau(i)} := V_{\chi^{-1}} \otimes V_i \cong V_i \otimes V_{\chi^{-1}}$ is simple as well. It is obvious that $i \in \Omega_0$ if and only if $\tau(i) \in \Omega_0$. Thus, the map τ gives a permutation on the index set Ω_0. In particular, one has $V_{\chi^{-1}} \otimes M(i, j) \cong M(\tau(i), j)$ for any $i \in \Omega_0$ and $1 \le j \le n$ (see [20, Proposition 3.1]).

Lemma 2.2 For any non-projective indecomposable module $M(i, j)$, $i \in \Omega_0$ and $1 \le j \le n-1$, we have the following:

(1) $\mathcal{N}(M(i, j)) \cong M(\tau^{1-n}(i), j)$.

(2) $\Omega(M(i, j)) \cong M(\tau^j(i), n-j)$ and $\Omega^{-1}(M(i, j)) \cong M(\tau^{j-n}(i), n-j)$.

Proof (1) Let $\psi := \mu \otimes \mathrm{id}$ be the automorphism of the algebra $H \otimes H^{\mathrm{op}}$. Consider $H^{(\psi)}$ the same as H as K-space while it is an $H \otimes H^{\mathrm{op}}$-module given by $(b \otimes c) \cdot x = \mu(b)xc$, for $b, c, x \in H$. Then the map

$$\Psi : H^{(\psi)} \to H^*$$

given by $\Psi(b)(c) = \langle b, c\rangle$ is bijective since the form $\langle -,-\rangle$ is non-degenerate. Moreover, the bijective above is an $H\otimes H^{\rm op}$-module isomorphism, where H^* is an $H\otimes H^{\rm op}$-module given by $((b\otimes c)\cdot g)(x) = g(cxb)$, for $b, c, x \in H$ and $g\in H^*$. In fact,

$$\begin{aligned}\Psi((b\otimes c)\cdot x)(z) &= \Psi(\mu(b)xc)(z) = \langle \mu(b)xc, z\rangle\\ &= \langle x, czb\rangle = \Psi(x)(czb)\\ &= ((b\otimes c)\cdot \Psi(x))(z)\end{aligned}$$

for any $b, c, x, z \in H$. As a result,

$$\mathcal{N}(M) \cong H^*\otimes_H M \cong H^{(\psi)}\otimes_H M \cong M^{(\mu)}$$

for any H-module M. For any non-projective indecomposable module $M(i, j)$, $i\in \Omega_0$ and $1\le j\le n-1$, by Lemma 2.1, the following linear map

$$M(i,j)^{(\mu)} \to V_{\chi^{n-1}}\otimes M(i,j), \quad x^k v\mapsto u\otimes x^k v$$

for any $v\in V_i$ and a fixed $0\neq u\in V_{\chi^{n-1}}$ is an H-module isomorphism. We conclude that $\mathcal{N}(M(i,j)) \cong V_{\chi^{n-1}}\otimes M(i,j)\cong M(\tau^{1-n}(i), j)$.

(2) Note that the epimorphism $p : M(i, n) \to M(i, j)$ given by

$$p\left(\sum_{k=0}^{n-1} x^k v_k\right) = \sum_{k=0}^{j-1} x^k v_k \quad \text{for } v_k\in V_i$$

induces an isomorphism $M(i,n)/\mathrm{rad}\, M(i,n) \cong M(i,j)/\mathrm{rad}\, M(i,j)$. The map p is a projective cover (see [1, Proposition 4.3, Chapter I]), it follows that $\Omega(M(i,j)) = \ker p$. It is straightforward to check that the map from $\Omega(M(i,j))$ to $M(\tau^j(i), n-j)$ given by

$$\sum_{k=j}^{n-1} x^k v_k \mapsto \sum_{k=j}^{n-1} x^{k-j}\sigma_{i,j}(v_k)$$

is an H-module isomorphism, where $\sigma_{i,j}$ is a bijective from V_i to $V_{\tau^j(i)}$ given in [20, Lemma 2.3]. We conclude that $\Omega(M(i,j))\cong M(\tau^j(i), n-j)$. The isomorphism $\Omega^{-1}(M(i,j))\cong M(\tau^{j-n}(i), n-j)$ follows immediately from the fact that $\Omega(M(i,j))\cong M(\tau^j(i), n-j)$. □

Remark 2.1 As mentioned earlier, the Auslander-Reiten translate $\mathrm{DTr}\cong \Omega^2\circ\mathcal{N}\cong\mathcal{N}\circ\Omega^2$. By Lemma 2.2, we obtain that $\mathrm{DTr}M(i,j)\cong M(\tau(i), j)$, which is exactly the result of [20, Proposition 3.2].

By the observation of Lemma 2.2, we have the following corollary:

Corollary 2.2 For any non-projective indecomposable module $M(i, j)$ and $m\in\mathbb{Z}$,

$$\Omega^m(M(i,j))\cong\begin{cases} M(\tau^{\frac{mn}{2}}(i), j), & 2\mid m,\\ M(\tau^{j+\frac{(m-1)n}{2}}(i), n-j), & 2\nmid m.\end{cases}$$

Recall that the stable category H-$\underline{\mathrm{mod}}$ is Calabi-Yau if and only if $\mathcal{N}\cong\Omega^{-(d+1)}$ of functors for some integer d. Denote by $\circ([1])$ the order of [1]. If $\circ([1]) = \infty$, then the integer d above is unique and is called the Calabi-Yau dimension of H. If $\circ([1])$ is finite, then the minimal non-negative integer d such that $\mathcal{N} \cong \Omega^{-(d+1)}$ is called the Calabi-Yau dimension of H (see e.g. [7, 9]).

Proposition 2.1 The stable category H-$\underline{\mathrm{mod}}$ is Calabi-Yau if and only if $n = 2$. In this case, the Calabi-Yau dimension of H is zero.

Proof We assume that H-$\underline{\mathrm{mod}}$ is Calabi-Yau, there is some integer d such that $\mathcal{N} \cong \Omega^{-(d+1)}$ and this isomorphism can be regarded as functors of H-module category (see [7]). For any non-projective indecomposable module $M(i,j)$, by Lemma 2.2 and Corollary 2.2, we have

$$\mathcal{N}(M(i,j)) \cong \Omega^{-(d+1)}(M(i,j))$$

if and only if

$$M(\tau^{1-n}(i),j) \cong \begin{cases} M(\tau^{-\frac{(d+1)n}{2}}(i),j), & 2 \mid d+1, \\ M(\tau^{j-\frac{(d+2)n}{2}}(i),n-j), & 2 \mid d. \end{cases}$$

In the case $2 \mid d+1$, the isomorphism $M(\tau^{1-n}(i),j) \cong M(\tau^{-\frac{(d+1)n}{2}}(i),j)$ implies that the order of τ divides $\frac{(d-1)n}{2}+1$. However, the order of τ is divisible by n. This yields that $n = 1$, a contradiction to Remark 1.1.

In the case $2 \mid d$, the isomorphism $M(\tau^{1-n}(i),j) \cong M(\tau^{j-\frac{(d+2)n}{2}}(i),n-j)$ implies that n is even, $j = \frac{n}{2}$ and the order of τ divides $\frac{(d-1)n}{2}+1$. Note that the order of τ is divisible by n. It follows that $n = 2$.

Conversely, if $n = 2$, then H is a Nakayama algebra of Loewy length 2. Hence, H-$\underline{\mathrm{mod}}$ is Calabi-Yau with the Calabi-Yau dimension zero (see [9, Proposition 2.1]). □

In the following, we shall determine all Calabi-Yau objects of H-$\underline{\mathrm{mod}}$ for the case $n > 2$. A d-th Calabi-Yau object M of H-$\underline{\mathrm{mod}}$ is said to be minimal if any proper direct summand of M is not a d-th Calabi-Yau object. Since every d-th Calabi-Yau object is a direct sum of finitely many minimal d-th Calabi-Yau objects [7, Theorem 4.2], we only need to describe all minimal Calabi-Yau objects of H-$\underline{\mathrm{mod}}$. By [7, Corollary 4.3], every minimal d-th Calabi-Yau object of H-$\underline{\mathrm{mod}}$ is of the form

$$\bigoplus_{0 \le k \le r_{ij}-1} (\Omega^{d+1} \circ \mathcal{N})^k (M(i,j)),$$

where r_{ij} is the relative order of $\Omega^{d+1} \circ \mathcal{N}$ with respect to non-projective indecomposable module $M(i,j)$. That is, r_{ij} is the minimal positive integer such that $(\Omega^{d+1} \circ \mathcal{N})^{r_{ij}}(M(i,j)) \cong M(i,j)$. It follows from Corollary 2.2 that

$$(\Omega^{d+1} \circ \mathcal{N})(M(i,j)) \cong \begin{cases} M(\tau^{1+\frac{(d-1)n}{2}}(i),j), & 2 \mid d+1, \\ M(\tau^{j+1+\frac{(d-2)n}{2}}(i),n-j), & 2 \mid d. \end{cases}$$

If $2 \mid d+1$, then for any $m \in \mathbb{Z}$, we have

$$(\Omega^{d+1} \circ \mathcal{N})^m (M(i,j)) \cong M(\tau^{m+\frac{(d-1)mn}{2}}(i),j).$$

If $2 \mid d$, then

$$(\Omega^{d+1} \circ \mathcal{N})^m (M(i,j)) \cong \begin{cases} M(\tau^{m+\frac{(d-1)mn}{2}}(i),j), & 2 \mid m, \\ M(\tau^{j+m+\frac{(d-1)mn-n}{2}}(i),n-j), & 2 \nmid m. \end{cases}$$

Together with Lemma 2.2, all minimal d-th Calabi-Yau objects of H-$\underline{\text{mod}}$ now can be completely described as follows:

Theorem 2.2 Let $n > 2$ and M be a minimal d-th Calabi-Yau object of H-$\underline{\text{mod}}$.

(1) If d is odd, then M is isomorphic to one of the following

$$\bigoplus_{0\leq m\leq m_i-1} M(\tau^{m+\frac{(d-1)mn}{2}}(i), j),$$

where $i \in \Omega_0$, $1 \leq j \leq n-1$ and m_i is the least positive integer satisfying $\tau^{m_i+\frac{(d-1)m_i n}{2}}(i) = i$.

(2) If d is even, then M is isomorphic to one of the following

$$\bigoplus_{\substack{0\leq m\leq m_i-1,\\ 2\mid m}} M(\tau^{m+\frac{(d-1)mn}{2}}(i), j) \oplus \bigoplus_{\substack{1\leq m\leq m_i,\\ 2\nmid m}} M(\tau^{j+m+\frac{(d-1)mn-n}{2}}(i), n-j),$$

where $i \in \Omega_0$, $1 \leq j \leq n-1$ and m_i is the least positive integer satisfying $\tau^{m_i+\frac{(d-1)m_i n}{2}}(i) = i$.

In particular, if the group datum $\mathcal{D}$ is of non-nilpotent type, then the order of τ is n, and the number n is the least positive integer satisfying $\tau^n(i) = i$ for any $i \in \Omega_0$ (see [21, Lemma 2.7 (5)]). In this case, by Theorem 2.2, the minimal d-th Calabi-Yau objects of H-$\underline{\text{mod}}$ can be described explicitly as follows:

Corollary 2.3 Let H be a finite dimensional pointed Hopf algebra of rank one associated to a group datum $\mathcal{D}$ of non-nilpotent type, $n > 2$ and M a minimal d-th Calabi-Yau object of H-$\underline{\text{mod}}$. If d is odd, then M is isomorphic to one of the following

$$\bigoplus_{0\leq k\leq n-1} M(\tau^k(i), j),$$

where $i \in \Omega_0$, $1 \leq j \leq n-1$. If d is even, then M is isomorphic to one of the following

(1)
$$\bigoplus_{0\leq k\leq \frac{n}{2}-1} (M(\tau^{2k}(i), j) \oplus M(\tau^{2k}(i), n-j)),$$
where n is even, $i \in \Omega_0$, $1 \leq j \leq n-1$, j is odd and $j \neq \frac{n}{2}$.

(2)
$$\bigoplus_{0\leq k\leq \frac{n}{2}-1} M(\tau^{2k}(i), j),$$
where n is even, $i \in \Omega_0$, $j = \frac{n}{2}$ is odd.

(3)
$$\bigoplus_{0\leq k\leq \frac{n}{2}-1} (M(\tau^{2k}(i), j) \oplus M(\tau^{2k+1}(i), n-j)),$$
where n is even, $i \in \Omega_0$, $1 \leq j \leq n-1$ and j is even.

(4)
$$\bigoplus_{0\leq k\leq n-1} (M(\tau^{k}(i), j) \oplus M(\tau^{k}(i), n-j)),$$
where n is odd, $i \in \Omega_0$, $1 \leq j \leq n-1$.

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秩为 1 的有限维 pointed Hopf 代数的稳定范畴中的 Calabi-Yau 对象

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摘要：设 H 是秩为 1 的有限维 pointed Hopf 代数. 借助于 Cibils 以及张的结果, 本文描述了 H 的稳定范畴 H-$\underline{\mathrm{mod}}$ 中的极小、以至所有 Calabi-Yau 对象.

关键词：稳定范畴; Calabi-Yau 范畴; Calabi-Yau 对象