doi: 10.11845/sxjz.2015174b

Positive Solutions for Second Order Three-point Boundary Value Problems With Sign-changing Nonlinearities

LI Hongyu*, ZHAO Mengtian

(*Department of Mathematics, Shandong University of Science and Technology, Qingdao, Shandong, 266590, P. R. China*)

Abstract: In this paper, by using topological degree methods and analyzing the boundary value conditions, we study the existence of positive solutions for second order three-point boundary value problems with sign-changing nonlinearity

Keywords: three-point boundary value problem; topological degree method; sign-changing nonlinearity

MR(2010) Subject Classification: 34B15; 47H11 / **CLC number:** O177.9; O175.1

Document code: A **Article ID:** 1000-0917(2017)05-0708-09

0 Introduction

In this paper, we shall establish the existence of positive solutions for the following second order three-point boundary value problem:

$$\varphi''(t) + a(t)\varphi'(t) + b(t)\varphi(t) = -g(t, \varphi(t)), \quad 0 \le t \le 1, \tag{0.1}$$

$$\varphi(0) = 0,\ \varphi(1) = \alpha\varphi(\eta), \tag{0.2}$$

where $0 < \alpha < 1,\ 0 < \eta < 1$.

The existence of nontrivial solutions for second order multi-point boundary value problems has been extensively studied by applying Krasnosel'skii's fixed point theorem, method of upper and lower solution, Leggett-Williams fixed point theorem, theory of fixed point index, and so on (see [1–17] and references therein). For example, in [6], by the fixed point theorems with lattice structure, under the condition that the nonlinear term can change sign, the author has studied the existence of nontrivial solutions of the following boundary value problems:

$$\begin{cases} -\varphi''(t) = g(t, \varphi(t)), \quad 0 \le t \le 1, \\ \varphi(0) = 0,\ \varphi(1) = \alpha\varphi(\eta), \end{cases}$$

where $0 < \alpha < 1,\ 0 < \eta < 1$, $g : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous.

In [3], by means of fixed point theorem, the authors considered the existence and uniqueness of positive solution for the following three-point boundary value problem with sign-changing

Received date: 2015-09-21.
Foundation item: This work is supported by NSFC (No. 11571207), the Research Award Fund for Outstanding Young Scientists of Shandong Province (No. BS2012SF022) and the Project of Shandong Province Higher Educational Science and Technology Program (No. J11LA07).
E-mail: ∗ sdlhy1978@163.com

nonlinearities:

$$\begin{cases} x''(t) + f(t, x(t)) = 0, & 0 < t < 1, \\ x(0) = 0, \ x(1) = \alpha x(\eta), & 0 < \eta < 1, \ 0 < \alpha < 1. \end{cases}$$

In [9], by using Leggett-Williams fixed point theorem, the authors considered the following second order three-point boundary value problem:

$$\begin{cases} u''(t) + a(t)u'(t) + f(t, u) = 0, & 0 \le t \le 1, \\ u'(0) = 0, \ u(1) = \alpha u(\eta), \end{cases}$$

where $0 < \alpha, \eta < 1$, $a : [0, 1] \to (-\infty, 0)$ is continuous, and f is allowed to change sign.

In [8], the authors considered the following second order three-point boundary value problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(t, u) = 0, & 0 < t < 1, \\ u(0) = 0, \ u(1) = \alpha u(\eta), \end{cases} \tag{0.3}$$

where α is a positive constant, $0 < \eta < 1$, $a \in C[0, 1]$, $b : [0, 1] \to (-\infty, 0)$ is continuous, $h : (0, 1) \to [0, \infty)$ is continuous, $f : [0, 1] \times (0, +\infty) \to [0, +\infty)$ is continuous, h is allowed to be singular at $t = 0, 1$, and f may be singular at $u = 0$. By the fixed point index theorem, the authors established the existence of positive solutions for the boundary value problem (0.3) under some conditions concerning the first eigenvalue corresponding to the relevant linear operator.

Inspired by [1–17], we shall discuss the existence of positive solutions and multiple positive solutions for the nonlinear second order three-point boundary value problem (0.1)–(0.2). The main features of this paper are as follows. First, there are few papers considering the boundary value problem (0.1)–(0.2) with sign-changing nonlinearity, i.e., when the nonlinear term g is allowed to change sign, we shall obtain the existence of one or two positive solutions for the boundary value problem (0.1)–(0.2) by using topological methods and analyzing the boundary value condition. Second, we shall give some relatively weak assumptions which are easy to be checked.

The organization of the rest of this paper is as follows. In Section 1, some preliminaries and lemmas are given, which will be used to prove the main results. In Section 2, we shall give the main results about the existence of positive solutions.

1 Preliminaries and Some Lemmas

In the following, we give the definition of H condition and some relevant lemmas. Let $E = C[0, 1]$ with the norm $\|\varphi\| = \max_{t\in[0,1]} |\varphi(t)|$. Then E is a Banach space. Let $P = \{\varphi \in E : \varphi(t) \ge 0, \ t \in [0, 1]\}$ be a cone of E.

Consider the linear integral operator

$$(B\varphi)(x) = \int_0^1 \overline{B}(x, y)\varphi(y)\mathrm{d}y, \tag{1.1}$$

where $\overline{B}(x, y)$ is nonnegative continuous on $[0, 1] \times [0, 1]$. It is obvious that $B : P \to P$ is completely continuous. Let $(B^*\psi)(x) = \int_0^1 \overline{B}^*(x, y)\psi(y)\mathrm{d}y$, where $\overline{B}^*(x, y) = \overline{B}(y, x)$. Let the spectral radius $r(B) \neq 0$. Then the spectral radius $r(B^*) = r(B)$. By Krein-Rutman theorem, there exists $\psi^* \in P \setminus \{\theta\}$ such that

$$\psi^* = \frac{1}{r(B)} B^* \psi^*. \tag{1.2}$$

For $\delta > 0$, let

$$P(\psi^*, \delta) = \left\{ \varphi \in P \middle| \int_0^1 \psi^*(x)\varphi(x)\mathrm{d}x \geq \delta\|\varphi\| \right\}.$$

Definition 1.1[1] Assume that there exists $\psi^* \in P \setminus \{\theta\}$ and $\delta > 0$ such that (1.2) is satisfied and B maps P into $P(\psi^*, \delta)$, then the operator B defined by (1.1) is said to satisfy H condition.

Further, let $\overline{B}(x,y) = k(x,y)a(y)$, $\overline{B}^*(x,y) = \overline{B}(y,x)$, where $k(x,y)$ is nonnegative continuous on $[0,1]\times[0,1]$, $a(y)$ is nonnegative continuous on $[0,1]$. Define

$$(B\varphi)(x) = \int_0^1 \overline{B}(x,y)\varphi(y)\mathrm{d}y, \tag{1.3}$$
$$(B^*\psi)(x) = \int_0^1 \overline{B}^*(x,y)\phi(y)\mathrm{d}y.$$

Lemma 1.1[1] Let $r(B) \neq 0$. And assume that there exists $v(x) \in P \setminus \{\theta\}$ such that

$$k(x,y) \geq v(x)k(\tau,y), \quad \forall x,y,\tau \in [0,1].$$

In addition, assume that there exists $\psi^*(x) \geq 0$, $\psi^*(x) \not\equiv 0$, $\psi^* = \frac{1}{r(B)}B^*\psi^*$, $v(x)\psi^*(x) \not\equiv 0$. Then the operator B defined by (1.3) satisfies H condition.

In order to prove our main results, we also need the following lemmas.

Lemma 1.2[1] Let E be an infinite Banach space, $\Omega \subset E$ be a bounded open set, $A : \overline{\Omega} \to E$ be completely continuous. And assume that $B : \partial\Omega \to E$ is completely continuous and satisfies

(i) $\inf_{x\in\partial\Omega} \|Bx\| > 0$,

(ii) $x - Ax \neq tBx$, $\forall x \in \partial\Omega$, $t \geq 0$.

Then $\deg(I - A, \Omega, 0) = 0$.

Lemma 1.3[1] Let E be a Banach space, $P \subset E$ be a cone, Ω be a bounded open set of E. And assume that $A : \overline{\Omega} \cap P \to P$ is completely continuous, and A has no fixed points on $P \cap \partial\Omega$. If there exist a linear operator $B : P \to P$ and $u_0 \in P \setminus \{\theta\}$ such that

(i) $B^n u_0 \geq u_0$, for some natural number n;

(ii) $Ax \geq Bx$, $\forall x \in P \cap \partial\Omega$.

Then $\deg(I - A, \Omega \cap P, 0; P) = 0$.

Lemma 1.4[10] Let $a \in C[0,1]$, $b \in C([0,1], (-\infty, 0))$. And assume that ϕ_1 is the unique solution of boundary value problem

$$\begin{cases} \phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t) = 0, & 0 < t < 1, \\ \phi_1(0) = 0,\ \phi_1(1) = 1, \end{cases}$$

and ϕ_2 is the unique solution of boundary value problem

$$\begin{cases} \phi_2''(t) + a(t)\phi_2'(t) + b(t)\phi_2(t) = 0, & 0 < t < 1, \\ \phi_2(0) = 1,\ \phi_2(1) = 0. \end{cases}$$

Then ϕ_1 is strictly increasing on $[0,1]$ and ϕ_2 is strictly decreasing on $[0,1]$.

Lemma 1.5[10] Let $a \in C[0,1]$, $b \in C([0,1],(-\infty,0))$. And assume that $0 < \alpha\phi_1(\eta) < 1$. Then for any $y \in C[0,1]$, the following boundary value problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \ u(1) = \alpha u(\eta) \end{cases}$$

is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s)p(s)y(s)u(s)\mathrm{d}s,$$

where

$$G(t,s) = k(t,s) + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)}k(\eta,s), \tag{1.4}$$

and

$$k(t,s) = \frac{1}{\rho}\begin{cases} \phi_1(t)\phi_2(s), \quad 0 \le t \le s \le 1, \\ \phi_1(s)\phi_2(t), \quad 0 \le s \le t \le 1, \end{cases}$$

and

$$\rho = \phi_1'(0) > 0, \quad p(t) = \exp\left(\int_0^t a(s)\mathrm{d}s\right).$$

For convenience, we list the following conditions.

(H_1) $a : [0,1] \to (-\infty,+\infty)$ and $b : [0,1] \to (-\infty,0)$ are continuous; $0 < \alpha\phi_1(\eta) < 1$, where ϕ_1 is given in Lemma 1.4.

(H_2) $g : [0,1] \times (-\infty,+\infty) \to (-\infty,+\infty)$ is continuous, $g(t,0) \equiv 0$ for $t \in [0,1]$, and there exists $m > 0$ such that $g(t,v) \ge -m$, $\forall t \in [0,1]$, $-\infty < v < +\infty$.

2 Main Results

Theorem 2.1 Suppose that (H_1) and (H_2) are satisfied. In addition, assume that there exist continuous functions $c(t) > 0$, $d(t) \ge 0$, $h(t) > 0$ and a constant $r > 0$ such that

$$g(t,v) \ge c(t)v - d(t), \quad \forall t \in [0,1], \ v \ge 0; \tag{2.1}$$

$$g(t,v) \le h(t)v, \quad \forall t \in [0,1], \ 0 \le v \le r. \tag{2.2}$$

And the spectral radius $r(L_\infty) > 1$, $r(L_0) \le 1$, where

$$(L_\infty\varphi)(t) = \int_0^1 G(t,s)p(s)c(s)\varphi(s)\mathrm{d}s, \tag{2.3}$$

$$(L_0\varphi)(t) = \int_0^1 G(t,s)p(s)h(s)\varphi(s)\mathrm{d}s.$$

Then the boundary value problem (0.1)–(0.2) has at least one positive solution.

In order to prove Theorem 2.1, we first need to prove the following lemma.

Lemma 2.1 The operator L_∞ defined by (2.3) satisfies H condition.

Proof Define

$$(L_\infty^*\varphi)(t) = \int_0^1 G^*(t,s)p(t)c(t)\varphi(s)\mathrm{d}s,$$

where $G^*(t,s)=G(s,t)$.

By the definition of $G^*(s,t)$, $p(t)$ and $c(t)$, there exists $t_0\in(0,1)$ such that $G^*(t_0,t_0)p(t_0)c(t_0)>0$. So there exists $[t_1,t_2]\subset[0,1]$ such that $G^*(t,s)p(t)c(t)>0$ for any $t,s\in[t_1,t_2]$. Hence, for any $t\in[t_1,t_2]$,

$$(L_\infty^*\varphi)(t)=\int_0^1 G^*(t,s)p(t)c(t)\varphi(s)\mathrm{d}s\geq\int_{t_1}^{t_2}G^*(t,s)p(t)c(t)\varphi(s)\mathrm{d}s>0.$$

Therefore, there exists $C>0$ such that $C(L_\infty^*\varphi)(t)\geq\varphi(t),\ t\in[0,1]$. By Krein-Rutman theorem, we obtain that the spectral radius $r(L_\infty^*)\neq 0$, and there exists $\omega^*(t)\in P\setminus\{\theta\}$ such that

$$L_\infty^*\omega^*=r(L_\infty^*)\omega^*. \tag{2.4}$$

By Lemma 1.4 and (1.4), we have

$$G(t,s)\geq\min\left\{\frac{\phi_1(t)}{\|\phi_1\|},\frac{\phi_2(t)}{\|\phi_2\|}\right\}G(\tau,s),\quad\forall t,s,\tau\in[0,1],$$

So

$$G(t,s)p(s)c(s)\geq\gamma(t)G(\tau,s)p(s)c(s),\quad\forall t,s,\tau\in[0,1].$$

where $\gamma(t)=\min\{\frac{\phi_1(t)}{\|\phi_1\|},\frac{\phi_2(t)}{\|\phi_2\|}\}$. Evidently, $\gamma(t)\omega^*(t)\not\equiv 0,\ \forall t\in[0,1]$.

Therefore, by Lemma 1.1, we get that Lemma 2.1 is valid. □

Proof of Theorem 2.1 Let

$$g_1(t,v)=\begin{cases}g(t,v), & v\geq 0,\\ 0, & v<0.\end{cases} \tag{2.5}$$

Consider the equation

$$\varphi''(t)+a(t)\varphi'(t)+b(t)\varphi(t)=-g_1(t,\varphi(t)),\quad 0\leq t\leq 1. \tag{2.6}$$

By (2.5), we obviously know that the positive solutions of the boundary value problem (2.6), (0.2) are the positive solutions of the boundary value problem (0.1)–(0.2).

By Lemma 1.5, the boundary value problem (2.6), (0.2) is equivalent to the following integral equation

$$\varphi(t)=\int_0^1 G(t,s)p(s)g_1(s,\varphi(s))\mathrm{d}s.$$

Define

$$(T\varphi)(t)=\int_0^1 G(t,s)p(s)g_1(s,\varphi(s))\mathrm{d}s,\quad\forall 0\leq t\leq 1. \tag{2.7}$$

By (H_1) and (H_2), $T,L_\infty,L_0:E\to E$ are completely continuous, and the fixed points of T defined by (2.7) are the solutions of the boundary value problem (2.6), (0.2).

From (2.1) and (H_2), there exists a constant $\widetilde{m}>0$ such that

$$\frac{g_1(t,v)}{c(t)}\geq-\widetilde{m},\quad\forall t\in[0,1],\ -\infty<v<+\infty. \tag{2.8}$$

By (2.1), (2.5) and (2.8), we easily get that there exists a continuous function $\widetilde{d}(t) \geq 0$ such that

$$g_1(t,v) \geq c(t)v - \widetilde{d}(t), \quad \forall t \in [0,1],\ -\infty < v < +\infty. \tag{2.9}$$

From Lemma 2.1, there exist $\phi^* \in P \setminus \{\theta\}$ and $\delta > 0$ such that $\phi^* = \frac{1}{r(L_\infty)} L_\infty^* \phi^*$ and L_∞ maps P into $P(\phi^*, \delta)$, where $P(\phi^*, \delta) = \{\varphi \in P : \int_0^1 \phi^*(t)\varphi(t)\mathrm{d}t \geq \delta\|\varphi\|\}$. Let $\beta = r(L_\infty) - 1$. Then $\beta > 0$.

Choose

$$R > \left\| \int_0^1 G(t,s)p(s)c(s)\widetilde{m}\mathrm{d}s \right\| + \frac{1}{\beta\delta}\left[\beta \int_0^1 \int_0^1 \phi^*(t)G(t,s)p(s)c(s)\widetilde{m}\mathrm{d}s\mathrm{d}t + \int_0^1 \int_0^1 \phi^*(t)G(t,s)p(s)\widetilde{d}(s)\mathrm{d}t\mathrm{d}s\right]. \tag{2.10}$$

Let $\widetilde{\varphi}(t)$ be a positive eigenfunction corresponding to the eigenvalue $r^{-1}(L_\infty)$. That is, $r(L_\infty)\widetilde{\varphi} = L_\infty\widetilde{\varphi}$. Let $\Omega_R = \{\varphi \in C[0,1] : \|\varphi\| < R\}$. We assume that there exist some $\overline{\varphi} \in \partial\Omega_R$ and $\mu_0 \geq 0$ such that

$$\overline{\varphi} - T\overline{\varphi} = \mu_0\widetilde{\varphi}. \tag{2.11}$$

Set $q(t) = \int_0^1 G(t,s)p(s)c(s)\widetilde{m}\mathrm{d}s$. Then, by (2.11) we get

$$\overline{\varphi}(t) + q(t) = (T\overline{\varphi})(t) + q(t) + \mu_0\widetilde{\varphi} = \int_0^1 G(t,s)p(s)c(s)\left[\frac{g_1(s,\overline{\varphi}(s))}{c(s)} + \widetilde{m}\right]\mathrm{d}s + \mu_0\widetilde{\varphi}. \tag{2.12}$$

Since $L_\infty : P \to P(\phi^*, \delta)$, we have $\overline{\varphi}(t) + q(t) \in P(\phi^*, \delta)$ by (2.12). Therefore, by (2.9)–(2.10) and (2.12), we obtain that

$$\begin{aligned}
&\int_0^1 \phi^*(t)T\overline{\varphi}(t)\mathrm{d}t - \int_0^1 \phi^*(t)\overline{\varphi}(t)\mathrm{d}t \\
&\geq \int_0^1 \phi^*(t)\mathrm{d}t \int_0^1 G(t,s)p(s)c(s)\overline{\varphi}(s)\mathrm{d}s - \int_0^1 \phi^*(t)\mathrm{d}t \int_0^1 G(t,s)p(s)\widetilde{d}(s)\mathrm{d}s - \int_0^1 \phi^*(t)\overline{\varphi}(t)\mathrm{d}t \\
&= r(L_\infty)\int_0^1 \phi^*(t)\overline{\varphi}(t)\mathrm{d}t - \int_0^1\int_0^1 \phi^*(t)G(t,s)p(s)\widetilde{d}(s)\mathrm{d}t\mathrm{d}s - \int_0^1 \phi^*(t)\overline{\varphi}(t)\mathrm{d}t \\
&= [r(L_\infty) - 1]\int_0^1 \phi^*(t)\overline{\varphi}(t)\mathrm{d}t - \int_0^1\int_0^1 \phi^*(t)G(t,s)p(s)\widetilde{d}(s)\mathrm{d}t\mathrm{d}s \\
&= \beta\int_0^1 \phi^*(t)\overline{\varphi}(t)\mathrm{d}t - \int_0^1\int_0^1 \phi^*(t)G(t,s)p(s)\widetilde{d}(s)\mathrm{d}t\mathrm{d}s \\
&= \beta\int_0^1 \phi^*(t)(\overline{\varphi}(t) + q(t))\mathrm{d}t - \beta\int_0^1 \phi^*(t)q(t)\mathrm{d}t - \int_0^1\int_0^1 \phi^*(t)G(t,s)p(s)\widetilde{d}(s)\mathrm{d}t\mathrm{d}s \\
&\geq \beta\delta\|\overline{\varphi} + q\| - \beta\int_0^1 \phi^*(t)q(t)\mathrm{d}t - \int_0^1\int_0^1 \phi^*(t)G(t,s)p(s)\widetilde{d}(s)\mathrm{d}t\mathrm{d}s \\
&\geq \beta\delta\|\overline{\varphi}\| - \beta\delta\|q\| - \beta\int_0^1 \phi^*(t)q(t)\mathrm{d}t - \int_0^1\int_0^1 \phi^*(t)G(t,s)p(s)\widetilde{d}(s)\mathrm{d}t\mathrm{d}s > 0.
\end{aligned} \tag{2.13}$$

On the other hand, by (2.11), we have

$$\int_0^1 \phi^*(t)\overline{\varphi}(t)\mathrm{d}t - \int_0^1 \phi^*(t)(T\overline{\varphi})(t)\mathrm{d}t = \mu_0\int_0^1 \phi^*(t)\widetilde{\varphi}(t)\mathrm{d}t \geq 0,$$

which contradicts with (2.13). So for any $\varphi \in \partial\Omega_R$, $\mu \geq 0$, we must have $\varphi - T\varphi \neq \mu\widetilde{\varphi}$. It follows from Lemma 1.2 that

$$\deg(I - T, \Omega_R, 0) = 0. \tag{2.14}$$

When R takes an arbitrary large number, (2.14) is valid. So by (2.14), we get that

$$\mathrm{ind}(I - T, \infty) = 0. \tag{2.15}$$

Let $\Omega_r = \{\varphi \in C[0,1] : \|\varphi\| < r\}$. Without loss of generality, we assume that T has no fixed points on $\partial\Omega_r$. In the following, we prove that for any $\varphi \in \partial\Omega_r,\ \zeta > 1$, we have

$$T\varphi \neq \zeta\varphi. \tag{2.16}$$

Otherwise, there exist $\varphi_0 \in \partial\Omega_r$ and $\zeta_0 > 1$ such that

$$T\varphi_0 = \zeta_0\varphi_0. \tag{2.17}$$

Now we claim that

$$\varphi_0(t) \geq 0, \quad \forall 0 \leq t \leq 1. \tag{2.18}$$

In fact, otherwise, we may assume that $\varphi_0(t)$ takes the minimum value at t_0. By (0.2), we know that $t_0 \in (0,1)$. So $\varphi_0(t_0) < 0$, $\varphi_0'(t_0) = 0$, $\varphi_0''(t_0) \geq 0$. Thus

$$\varphi_0''(t_0) + a(t_0)\varphi_0'(t_0) + b(t_0)\varphi_0(t_0) > 0. \tag{2.19}$$

But $\varphi_0''(t_0) + a(t_0)\varphi_0'(t_0) + b(t_0)\varphi_0(t_0) = -{\zeta_0}^{-1}g_1(t_0, \varphi_0(t_0)) = 0$, which contradicts with (2.19). So (2.18) holds.

By (2.2) and (2.17), we have

$$\begin{aligned}\zeta_0\varphi_0(t) = (T\varphi_0)(t) &= \int_0^1 G(t,s)p(s)g_1(s,\varphi_0(s))\mathrm{d}s = \int_0^1 G(t,s)p(s)g(s,\varphi_0(s))\mathrm{d}s \\ &\leq \int_0^1 G(t,s)p(s)h(s)\varphi_0(s)\mathrm{d}s = (L_0\varphi_0)(t).\end{aligned} \tag{2.20}$$

By (2.20) and Krein-Rutman theorem, $r(L_0) \geq \zeta_0 > 1$, which contradicts with $r(L_0) \leq 1$. Therefore, (2.16) holds. So we have

$$\deg(I - T, \Omega_r, 0) = 1. \tag{2.21}$$

By (2.15), (2.21) and the property of topological degree, there exists $\varphi_*(t) \in C[0,1],\ \varphi_*(t) \not\equiv 0$ such that $\varphi_*(t) = (T\varphi_*)(t)$. Similar to the proof of (2.18), we get that $\varphi_*(t) \geq 0,\ t \in [0,1]$. So $\varphi_*(t)$ is a positive solution of the boundary value problem (2.6), (0.2). By (2.5), we know that $\varphi_*(t)$ is the positive solution of the boundary value problem (0.1)–(0.2). □

Theorem 2.2 Suppose that $(\mathrm{H}_1)-(\mathrm{H}_2)$ and (2.1) of Theorem 2.1 are satisfied. In addition, assume that there exist $r^* > r_1 > 0$ and a continuous function $w(t) \geq 0$ such that

$$g(t,v) \geq w(t)v, \quad \forall t \in [0,1],\ 0 \leq v \leq r_1. \tag{2.22}$$

$$g(t,r^*) < 0, \quad \forall t \in [0,1]. \tag{2.23}$$

And $r(L_\infty) > 1,\ r(L_1) \geq 1$, where L_∞ is defined by (2.3), L_1 is defined by

$$(L_1\varphi)(t) = \int_0^1 G(t,s)p(s)w(s)\varphi(s)\mathrm{d}s.$$

Then the boundary value problem (0.1)–(0.2) has at least two positive solutions.

Proof By the proof of Theorem 2.1, we only need to prove that the boundary value problem (2.6), (0.2) has two nontrivial solutions. Obviously, (2.15) holds.

Let $\Omega_{r^*} = \{\varphi \in C[0,1] : \|\varphi\| < r^*\}$. Without loss of generality, we assume that T has no fixed points on $\partial\Omega_r^*$. Assume that there exist $\varphi_1 \in \partial\Omega_{r^*}$ and $\xi_1 > 1$ such that

$$T\varphi_1 = \xi_1\varphi_1. \tag{2.24}$$

Similar to the proof of (2.18), we know that

$$\varphi_1(t) \geq 0, \quad \forall t \in [0,1]. \tag{2.25}$$

Since $\|\varphi_1\| = r^*$ and (0.2), there exists $t_1 \in (0,1)$ such that $\varphi_1(t_1) = r^*$. Then $\varphi_1'(t_1) = 0$, $\varphi_1''(t_1) \leq 0$. So

$$\varphi_1''(t_1) + a(t_1)\varphi_1'(t_1) + b(t_1)\varphi_1(t_1) \leq 0. \tag{2.26}$$

By (2.23)–(2.24), we have

$$\varphi_1''(t_1) + a(t_1)\varphi_1'(t_1) + b(t_1)\varphi_1(t_1) = -\xi_1^{-1}g_1(t_1,\varphi_1(t_1)) = -\xi_1^{-1}g(t_1,r^*) > 0,$$

which contradicts with (2.26). So (2.25) holds. Hence, for any $\varphi \in \partial\Omega_{r^*}$, $\xi \geq 1$, $T\varphi \neq \xi\varphi$. Then

$$\deg(I - T, \Omega_{r^*}, 0) = 1. \tag{2.27}$$

Let $\Omega_{r_1} = \{\varphi \in C[0,1] : \|\varphi\| < r_1\}$. Let $P = \{\varphi \in C[0,1] : \varphi(t) \geq 0\}$. Then P is a cone of $C[0,1]$. By the definition of T and (2.5), (2.22), T maps $\overline{\Omega}_{r_1}$ into P. By the maintenance of topological degree, we have

$$\deg(I - T, \Omega_{r_1}, 0) = \deg(I - T, \Omega_{r_1} \cap P, 0; P). \tag{2.28}$$

By (2.22), for any $\varphi \in \partial\Omega_{r_1} \cap P$, we get

$$(T\varphi)(t) = \int_0^1 G(t,s)p(s)g_1(s,\varphi(s))\mathrm{d}s \geq \int_0^1 G(t,s)p(s)w(s)\varphi(s)\mathrm{d}s = (L_1\varphi)(t). \tag{2.29}$$

Since $r(L_1) \geq 1$, there exists $\widetilde{\varphi} \in P \setminus \{\theta\}$ such that $L_1\widetilde{\varphi} = r(L_1)\widetilde{\varphi} \geq \widetilde{\varphi}$. So by (2.29) and Lemma 1.3, we have

$$\deg(I - T, \Omega_{r_1} \cap P, 0; P) = 0. \tag{2.30}$$

From (2.27) and (2.30), we have

$$\deg(I - T, \Omega_{r_1}, 0) = 0. \tag{2.31}$$

By (2.14), (2.27)–(2.28) and (2.31), T has at least two fixed points. Hence, the boundary value problem (2.6), (0.2) has at least two nontrivial solutions. Similar to the proof of (2.17), we know that the two nontrivial solutions are positive solutions. From (2.5), the boundary value problem (0.1)–(0.2) has at least two positive solutions. □

References

[1] Guo, D.J. and Sun, J.X., Nonlinear Integral Equations, Ji'nan: Shandong Science and Technology Press, 1987 (in Chinese).

[2] Ji, C.S., O'Regan, D., Yan, B.Q. and Agarwal, R.P., Nonexistence and existence of positive solutions for second order singular three-point boundary value problems with derivative dependent and sign-changing nonlinearities, *J. Appl. Math. Comput.*, 2011, 36(1/2): 61-87.

[3] Ji, C.S. and Yan, B.Q., Positive solutions for second-order singular three-point boundary value problems with sign-changing nonlinearities, *Electron. J. Differential Equations.*, 2010, 2010(38): 9 pages.

[4] Lan, K.Q., Eigenvalues of semi-positone Hammerstein integral equations and applications to boundary value problems, *Nonlinear Anal.*, 2009, 71(12): 5979-5993.

[5] Li, G.S., Liu, X.P. and Jia, M., Positive solutions to a type of nonlinear three-point boundary value problem with sign changing nonlinearities, *Comput. Math. Appl.*, 2009, 57(3): 348-355.

[6] Li, H.Y., Existence of nontrivial solutions for unilaterally asymptotically linear three-point boundary value problems, *Abstr. Appl. Anal.*, 2014, 2014: Article ID 263042, 7 pages.

[7] Li, J.L. and Shen, J.H., Multiple positive solutions for a second-order three-point boundary value problem, *Appl. Math. Comput.*, 2006, 182(1): 258-268.

[8] Liu, B.M., Liu, L.S. and Wu, Y.H., Positive solutions for a singular second-order three-point boundary value problems, *Appl. Math. Comput.*, 2008, 196(2): 532-541.

[9] Liu, J. and Zhao, Z.Q., Multiple positive solutions for second-order three-point boundary value problems with sign changing nonlinearities, *Electron. J. Differential Equations*, 2012, 2012(151): 1-7.

[10] Ma, R.Y. and Wang, H.Y., Positive solutions of nonlinear three-point boundary value problems, *J. Math. Anal. Appl.*, 2003, 279(1): 216-227.

[11] Sun, Y.P. and Zhao, M., Existence of positive pseudo-symmetric solution for second-order three-point boundary value problems, *J. Appl. Math. Comput.*, 2015, 47(1/2): 211-224.

[12] Xu, F.Y., Liu, L.S. and Meng, Z.W., Positive solutions for nonlinear second-order m-point boundary value problems with sign changing nonlinearities, *Nonlinear Funct. Anal. Appl.*, 2009, 14(2): 181-193.

[13] Xu, X., Multiplicity results for positive solutions of some semi-positone three-point boundary value problems, *J. Math. Anal. Appl.*, 2004, 291(2): 673-689.

[14] Xu, X., O'Regan, D. and Sun, J.X., Multiplicity results for three-point boundary value problems with a non-well-ordered upper and lower solution condition, *Math. Comput. Modelling*, 2007, 45(1/2): 189-200.

[15] Zhai, C.B., Positive solutions for semi-positone three-point boundary value problems, *J. Comput. Appl. Math.*, 2009, 228(1): 279-286.

[16] Zhang, G.W. and Sun, J.X., Positive solutions of m-point boundary value problems, *J. Math. Anal. Appl.*, 2004, 291(2): 406-418.

[17] Zhong, X.C. and Huang, Q.A., Approximate solution of three-point boundary value problems for second-order ordinary differential equations with variable coefficients, *Appl. Math. Comput.*, 2014, 247: 18-29.

具有变号非线性项的二阶三点边值问题的正解

李红玉， 赵梦田

(山东科技大学数学系, 青岛, 山东, 266590)

摘要： 利用拓扑度方法, 结合分析边值条件, 研究了一类具有变号非线性项的二阶三点边值问题的正解的存在性.

关键词： 三点边值问题; 拓扑度方法; 变号非线性项