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Spectra of Partially Subdivision Neighbourhood Coronae

ZHU Xueqin[1], TIAN Guixian[1,*], CUI Shuyu[2]

(*1. College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua, Zhejiang, 321004, P. R. China; 2. Xingzhi College, Zhejiang Normal University, Jinhua, Zhejiang, 321004, P. R. China*)

Abstract: Let G_1, G_2 be two simple connected graphs. The partially subdivision neighbourhood corona of G_1 and G_2, denoted by $G_1 \overline{\star} G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining the neighbours of the i-th vertex of G_1 to every vertex in the i-th copy of G_2, then inserting a new vertex into every edge of G_1. In this paper, we determine the adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of $G_1 \overline{\star} G_2$ in terms of those of two factor graphs G_1 and G_2. In addition, as many applications of these results, we consider constructing infinite pairs of adjacency cospectral, Laplacian cospectral and signless Laplacian cospectral graphs. Moreover, we compute the number of spanning trees of $G_1 \overline{\star} G_2$ in terms of the Laplacian spectra of two factor graphs G_1 and G_2.

Keywords: adjacency matrix; Laplacian matrix; signless Laplacian matrix; spectrum of a graph; partially subdivision neighbourhood corona

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0 Introduction

All graphs considered in this paper are finite, simple connected graphs. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G, denoted by $A(G)$, is an $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. The degree of v_i in G is denoted by $d_i = d_G(v_i)$. Let $D(G)$ be the diagonal degree matrix of G which diagonal entries are $d_1, d_2, \cdots, d_n$. The *Laplacian matrix* $L(G)$ of G is defined as $D(G) - A(G)$. The *signless Laplacian matrix* of G is defined as $Q(G) = D(G) + A(G)$. For an $n \times n$ matrix M associated to G, the characteristic polynomial $\det(xI_n - M)$ of M is called the M-*characteristic polynomial* of G and is denoted by $\phi(M; x)$. The eigenvalues of M (i.e., the zeros of $\det(xI_n - M)$) and the spectrum of M (which consists of the n eigenvalues) are also called the M-*eigenvalues* of G and the M-*spectrum* of G, respectively. In particular, if M is the adjacency matrix $A(G)$ of G, then the A-spectrum of G is denoted by $\sigma(A(G)) = (\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G))$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ are the eigenvalues of $A(G)$. If M is the Laplacian matrix $L(G)$ of G, then the L-spectrum of G is denoted by $\sigma(L(G)) = (\mu_1(G), \mu_2(G), \cdots, \mu_n(G))$, where $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ are the eigenvalues of $L(G)$. If M is the signless Laplacian matrix

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E-mail: * gxtian@zjnu.cn

$Q(G)$ of G, then the Q-spectrum of G is denoted by $\sigma(Q(G)) = (q_1(G), q_2(G), \cdots, q_n(G))$, where $q_1(G) \le q_2(G) \le \cdots \le q_n(G)$ are the eigenvalues of $Q(G)$. We call that two graphs G_1 and G_2 are A-cospectral (resp., L-cospectral, Q-cospectral) whenever they have the same A-spectrum (resp., L-spectrum and Q-spectrum). For more review about the A-spectrum, L-spectrum and Q-spectrum of G, readers may refer to [4–6, 8] and the references therein.

It is of interest to study some spectral properties of certain composite operations between two graphs, such as the Cartesian product, the Kronecker product, the corona, the edge corona and the neighbourhood corona. For example, the A-spectra, L-spectra and Q-spectra of the (edge) corona of two graphs can be expressed by these of the two factor graphs[1−3, 10, 14−15]. Recently, the neighbourhood corona of two graphs was defined in [7] and the A-spectra, L-spectra and Q-spectra of the neighbourhood corona of two graphs were computed in [7, 12]. The subdivision graph $S(G)$ (see [6]) of a graph G is the graph obtained by inserting a new vertex into each edge of G. Based on subdivision graphs, the subdivision-vertex and subdivision-edge (neighbourhood) coronae were introduced and their A-spectra, L-spectra and Q-spectra were also given in terms of these of two factor graphs in [11, 13], respectively. Motivated by the works above, we define a new graph operation based on subdivision graphs as follows.

Definition 0.1 The partially subdivision neighbourhood corona of two graphs G_1 and G_2, denoted by $G_1 \barwedge G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining the neighbours of the i-th vertex of G_1 to every vertex in the i-th copy of G_2, then inserting a new vertex into every edge of G_1.

Note that if G_1 and G_2 are two graphs on disjoint vertex sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively, then the partially subdivision neighbourhood corona $G_1 \barwedge G_2$ has $n_1 + m_1 + n_1 n_2$ vertices and $n_1 m_2 + 2m_1(1 + n_2)$ edges.

Example 0.1 Let P_4 and P_2 be two paths of 4 and 2 vertices, respectively. The partially subdivision neighbourhood corona $P_4 \barwedge P_2$ of P_4 and P_2 is depicted in Figure 1.

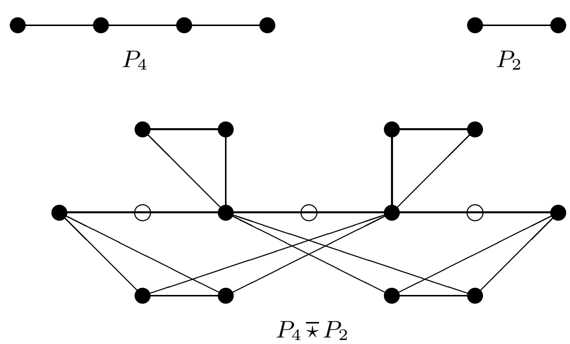

Figure 1 An example of partially subdivision neighbourhood corona

In this paper, we discuss the A-spectra, L-spectra and Q-spectra of partially subdivision neighbourhood corona $G_1 \barwedge G_2$ for two graphs G_1 and G_2. First, we compute the A-characteristic polynomial and Q-characteristic polynomial of $G_1 \barwedge G_2$ for a regular graph G_1 and an arbitrary graph G_2. Using these results, we give a complete description of the A-spectra and Q-spectra of $G_1 \overline{\star} G_2$ whenever G_1 is an r_1-regular graph and G_2 is an r_2-regular (or complete bipartite) graph. Second, we compute the L-characteristic polynomial of $G_1 \overline{\star} G_2$ and give a complete description of its L-spectra for a regular graph G_1 and an arbitrary graph G_2. Finally, as many applications of these results, we consider constructing infinite pairs of A-cospectral, L-cospectral and Q-cospectral graphs. Moreover, we compute the number of spanning trees of $G_1 \overline{\star} G_2$ in terms of the L-spectra of two factor graphs G_1 and G_2.

1 Spectra of Partially Subdivision Neighbourhood Coronae

In this section, we discuss the A-spectra, L-spectra and Q-spectra of partially subdivision neighbourhood coronae with the help of the *coronal* of a matrix. The M-coronal $\Gamma_M(x)$ of a matrix M of order n is defined[3, 14] to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is, $\Gamma_M(x) = 1_n^{\mathrm{T}}(xI_n - M)^{-1}1_n$, where 1_n denotes the column vector of size n with all the entries equal to 1.

The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. It is well known that[9] $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$, whenever the products AC and BD exist. Moreover, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for two non-singular matrices A and B. If A and B are two matrices of order n and p respectively, then $\det(A \otimes B) = (\det A)^p(\det B)^n$. Other properties of the Kronecker product can be found in [9].

Given two arbitrary graphs G_1 and G_2 of order n_1 and n_2, m_1 and m_2 edges, respectively, we first label the vertices of $G_1 \overline{\star} G_2$ as follows. Let $V(G_1) = \{v_1, v_2, \cdots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \cdots, e_{m_1}\}$, and $V(G_2) = \{u_1, u_2, \cdots, u_{n_2}\}$. For $i = 1, 2, \cdots, n_1$, let $V^i(G_2) = \{u_1^i, u_2^i, \cdots, u_{n_2}^i\}$ denote the vertex set of the i-th copy of G_2. Then $V(G_1) \cup I(G_1) \cup [\bigcup_{i=1}^{n_1} V^i(G_2)]$ is a partition of $V(G_1 \overline{\star} G_2)$. Moreover, the degrees of the vertices of $G_1 \overline{\star} G_2$ are

$$d_{G_1 \overline{\star} G_2}(v_i) = (n_2+1)d_{G_1}(v_i), \quad i = 1, 2, \cdots, n_1;$$
$$d_{G_1 \overline{\star} G_2}(u_j^i) = d_{G_2}(u_j) + d_{G_1}(v_i), \quad i = 1, 2, \cdots, n_1,\ j = 1, 2, \cdots, n_2;$$
$$d_{G_1 \overline{\star} G_2}(e_i) = 2, \quad i = 1, 2, \cdots, m_1.$$

1.1 A-spectra of $G_1 \overline{\star} G_2$

Theorem 1.1 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an arbitrary graph on n_2 vertices. Then

$$\phi(A(G_1 \overline{\star} G_2); x) = x^{m_1 - n_1}(\phi(A(G_2)))^{n_1} \prod_{i=1}^{n_1} \left[(x^2 - r_1) - \lambda_i(G_1) - x\Gamma_{A(G_2)}(x)\lambda_i^2(G_1)\right].$$

Proof With respect to the partition $V(G_1) \cup I(G_1) \cup [\bigcup_{i=1}^{n_1} V^i(G_2)]$ of $V(G_1 \overline{\star} G_2)$, we can write the adjacency matrix of $G_1 \overline{\star} G_2$ as

$$A(G_1 \overline{\star} G_2) = \begin{pmatrix} 0 & R & 1_{n_2}^{\mathrm{T}} \otimes A(G_1) \\ R^{\mathrm{T}} & 0 & 0 \\ 1_{n_2} \otimes A(G_1) & 0 & A(G_2) \otimes I_{n_1} \end{pmatrix},$$

where $R = (r_{ij})$ is the vertex-edge incidence matrix of G_1 with entry $r_{ij} = 1$ if the vertex i is incident to the edge e_j and 0 otherwise. Then the A-characteristic polynomial of $G_1 \overline{\star} G_2$ is given by

$$\begin{aligned}\phi(A(G_1 \overline{\star} G_2)) &= \det \begin{pmatrix} xI_{n_1} & -R & -1_{n_2}^{\mathrm{T}} \otimes A(G_1) \\ -R^{\mathrm{T}} & xI_{m_1} & 0 \\ -1_{n_2} \otimes A(G_1) & 0 & xI_{n_1 n_2} - A(G_2) \otimes I_{n_1} \end{pmatrix} \\ &= \det((xI_{n_2} - A(G_2)) \otimes I_{n_1}) \det(S) \\ &= (\phi(A(G_2)))^{n_1} \det(S),\end{aligned}$$

where

$$\begin{aligned}S &= \begin{pmatrix} xI_{n_1} & -R \\ -R^{\mathrm{T}} & xI_{m_1} \end{pmatrix} - \begin{pmatrix} -1_{n_2}^{\mathrm{T}} \otimes A(G_1) \\ 0 \end{pmatrix} ((xI_{n_2} - A(G_2)) \otimes I_{n_1})^{-1} \begin{pmatrix} -1_{n_2} \otimes A(G_1) & 0 \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} & -R \\ -R^{\mathrm{T}} & xI_{m_1} \end{pmatrix} - \begin{pmatrix} \Gamma_{A(G_2)}(x) A^2(G_1) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - \Gamma_{A(G_2)}(x) A^2(G_1) & -R \\ -R^{\mathrm{T}} & xI_{m_1} \end{pmatrix}\end{aligned}$$

is the Schur complement[16] of $(xI_{n_2} - A(G_2)) \otimes I_{n_1}$ and

$$\begin{aligned}\det(S) &= \det(xI_{m_1}) \det(xI_{n_1} - \Gamma_{A(G_2)}(x) A^2(G_1) - (-R)(xI_{m_1})^{-1}(-R^{\mathrm{T}})) \\ &= x^{m_1} \det\left[xI_{n_1} - \Gamma_{A(G_2)}(x) A^2(G_1) - \frac{1}{x}(r_1 I_{n_1} + A(G_1))\right] \\ &= x^{m_1} \det\left[\left(x - \frac{1}{x} r_1\right) I_{n_1} - \Gamma_{A(G_2)}(x) A^2(G_1) - \frac{1}{x} A(G_1)\right] \\ &= x^{m_1 - n_1} \prod_{i=1}^{n_1} [(x^2 - r_1) - \lambda_i(G_1) - x\Gamma_{A(G_2)}(x) \lambda_i^2(G_1)],\end{aligned}$$

where $RR^{\mathrm{T}} = r_1 I_{n_1} + A(G_1)$ and $\lambda_i(G_1)$ is the i-th eigenvalue of G_1. □

Corollary 1.1 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the A-spectrum of $G_1 \overline{\star} G_2$ consists of

(i) 0, repeated $m_1 - n_1$ times;

(ii) $\lambda_i(G_2)$, repeated n_1 times for $i = 2, 3, \cdots, n_2$;

(iii) three roots of the equation

$$x^3 - r_2 x^2 - (r_1 + \lambda_j(G_1) + n_2 \lambda_j^2(G_1))x + r_1 r_2 + \lambda_j(G_1) r_2 = 0$$

for each $j = 1, 2, \cdots, n_1$.

Proof Since G_2 is r_2-regular, then by [3, Proposition 2],

$$\Gamma_{A(G_2)}(x) = \frac{n_2}{x - r_2}.$$

The only pole of $\Gamma_{A(G_2)}(x)$ is the maximal eigenvalue $x = r_2$ of G_2. Thus, by Theorem 1.1, $\lambda_i(G_2)$ is an eigenvalue of $G_1 \overline{\star} G_2$ repeated n_1 times for each $i = 2, 3, \cdots, n_2$ and 0 is also an eigenvalue of $G_1 \overline{\star} G_2$ repeated $m_1 - n_1$ times. The remaining eigenvalues are obtained by solving

$$x^2 - r_1 - \lambda_j(G_1) - \frac{n_2 x}{x - r_2} \lambda_j^2(G_1)$$

for each $j = 1, 2, \cdots, n_1$. Hence, the result follows. □

Corollary 1.2 Let G be an r-regular graph on n vertices and m edges, and $K_{p,q}$ be a complete bipartite graph with $p, q \geq 1$. Then the A-spectrum of $G \barwedge K_{p,q}$ consists of

(i) 0, repeated $(p+q-3)n+m$ times;

(ii) four roots of the equation

$$x^4 - [pq + \lambda_j^2(G)(p+q) + r + \lambda_j(G)]x^2 - 2pq\lambda_j^2(G)x + pq(r + \lambda_j(G)) = 0 \quad \text{for } j = 1, 2, \cdots, n.$$

Proof By [14, Proposition 8], we have

$$\Gamma_{A(K_{p,q})}(x) = \frac{(p+q)x + 2pq}{x^2 - pq}.$$

Note that the A-spectrum of $G_2 = K_{p,q}$ is $\sigma(A(K_{p,q})) = (0^{(p+q-2)}, \pm\sqrt{pq})$. The two poles of $\Gamma_{A(K_{p,q})}(x)$ are the non-zero eigenvalues $x = \pm\sqrt{pq}$ of $K_{p,q}$. Theorem 1.1 implies the required result immediately. □

It is well known that[1, 3, 11–14] many infinite families of pairs of A-cospectral graphs are generated by using graph operations. As an application of Theorem 1.1, we also consider constructing infinite pairs of A-cospectral graphs by employing the partially subdivision neighbourhood corona of two graphs.

Corollary 1.3 (i) If G_1 and G_2 are two A-cospectral r-regular graphs, and H is an arbitrary graph, then $G_1 \barwedge H$ and $G_2 \barwedge H$ are A-cospectral;

(ii) If G is a regular graph, H_1 and H_2 are two A-cospectral graphs with $\Gamma_{A(H_1)}(x) = \Gamma_{A(H_2)}(x)$, then $G \barwedge H_1$ and $G \barwedge H_2$ are A-cospectral.

1.2 Q-spectra of $G_1 \barwedge G_2$

Theorem 1.2 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an arbitrary graph on n_2 vertices. Then

$$\begin{aligned}\phi(Q(G_1 \barwedge G_2); x) = {} & (x-2)^{m_1} \prod_{i=1}^{n_2} (x - r_1 - q_i(G_2))^{n_1} \\ & \cdot \prod_{i=1}^{n_1} \left[x - r_1(n_2+1) - \Gamma_{Q(G_2)}(x - r_1)\lambda_i^2(G_1) - \frac{r_1 + \lambda_i(G_1)}{x-2} \right].\end{aligned}$$

Proof With respect to the partition $V(G_1) \cup I(G_1) \cup [\bigcup_{i=1}^{n_1} V^i(G_2)]$ of $V(G_1 \barwedge G_2)$, the degree diagonal matrix of $G_1 \barwedge G_2$ can be written as

$$D(G_1 \barwedge G_2) = \begin{pmatrix} (n_2+1)r_1 I_{n_1} & 0 & 0 \\ 0 & 2I_{m_1} & 0 \\ 0 & 0 & (D(G_2) + r_1 I_{n_2}) \otimes I_{n_1} \end{pmatrix}.$$

Thus,

$$Q(G_1 \barwedge G_2) = \begin{pmatrix} (n_2+1)r_1 I_{n_1} & R & 1_{n_2}^{\mathrm{T}} \otimes A(G_1) \\ R^{\mathrm{T}} & 2I_{m_1} & 0 \\ 1_{n_2} \otimes A(G_1) & 0 & (Q(G_2) + r_1 I_{n_2}) \otimes I_{n_1} \end{pmatrix}.$$

Then the Q-characteristic polynomial of $G_1 \overline{\star} G_2$ is given by

$$\begin{aligned}\phi(Q(G_1 \overline{\star} G_2)) &= \det\begin{pmatrix} (x-(n_2+1)r_1)I_{n_1} & -R & -1_{n_2}^{\mathrm{T}} \otimes A(G_1) \\ -R^{\mathrm{T}} & (x-2)I_{m_1} & 0 \\ -1_{n_2} \otimes A(G_1) & 0 & ((x-r_1)I_{n_2} - Q(G_2)) \otimes I_{n_1} \end{pmatrix} \\ &= \det[((x-r_1)I_{n_2} - Q(G_2)) \otimes I_{n_1}] \cdot \det(S) \\ &= \prod_{i=1}^{n_2} (x - r_1 - q_i(G_2))^{n_1} \cdot \det(S),\end{aligned}$$

where

$$S = \begin{pmatrix} (x-(n_2+1)r_1)I_{n_1} - \Gamma_{Q(G_2)}(x-r_1)A^2(G_1) & -R \\ -R^{\mathrm{T}} & (x-2)I_{m_1} \end{pmatrix}$$

is the Schur complement[16] of $((x-r_1)I_{n_2} - Q(G_2)) \otimes I_{n_1}$ and

$$\begin{aligned}\det(S) &= \det((x-2)I_{m_1}) \cdot \det\left[(x-(n_2+1)r_1)I_{n_1} - \Gamma_{Q(G_2)}(x-r_1)A^2(G_1) - \frac{RR^{\mathrm{T}}}{x-2}\right] \\ &= (x-2)^{m_1} \prod_{i=1}^{n_1} \left[x - r_1(n_2+1) - \Gamma_{Q(G_2)}(x-r_1)\lambda_i^2(G_1) - \frac{r_1 + \lambda_i(G_1)}{x-2}\right],\end{aligned}$$

where $RR^{\mathrm{T}} = r_1 I_{n_1} + A(G_1)$ and $\lambda_i(G_1)$ is the i-th eigenvalue of G_1. □

Corollary 1.4 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the Q-spectrum of $G_1 \overline{\star} G_2$ consists of

(i) $r_1 + q_i(G_2)$, repeated n_1 times for each $i = 1, 2, \cdots, n_2 - 1$;

(ii) 2, repeated $m_1 - n_1$ times;

(iii) three roots of the equation, for each $j = 1, 2, \cdots, n_1$,

$$x^3 - [(n_2+2)r_1 + 2r_2 + 2]x^2 + a_j x + b_j = 0,$$

where

$$\begin{aligned}a_j &= [(n_2+1)r_1^2 + (2n_2+2)r_1 r_2 + (n_2+2)r_1 + 4r_2 - n_2\lambda_j^2(G_1) - \lambda_j(G_1)], \\ b_j &= (r_1 + 2r_2)(\lambda_j(G_1) - 2r_1 n_2 - r_1) + 2n_2\lambda_j^2(G_1).\end{aligned}$$

Proof Since G_2 is r_2-regular, then by [3, Proposition 2],

$$\Gamma_{Q(G_2)}(x - r_1) = \frac{n_2}{x - r_1 - 2r_2}.$$

Thus, by Theorem 1.2, we obtain the required result immediately. □

Corollary 1.5 Let G be an r-regular graph on n vertices and m edges, and $K_{p,q}$ be a complete bipartite graph with $p, q \geq 1$. Then the Q-spectrum of $G \overline{\star} K_{p,q}$ consists of

(i) $p + r$, repeated $n(q-1)$ times;

(ii) $q + r$, repeated $n(p-1)$ times;

(iii) 2, repeated $m - n$ times;

(iv) four roots of the equation, for each $j = 1, 2, \cdots, n$,

$$x^4 - [(p+q)(r+1) + 3r + 2]x^3 + a_j x^2 + b_j x + c_j = 0,$$

where

$$\begin{aligned}a_j &= r(p+q)^2 + (2r^2+4r+2-\lambda_j^2(G))(p+q) + 3r^2 + 5r - \lambda_j(G),\\ b_j &= -(r^2+2r)(p+q)^2 - [r^3+5r^2-(\lambda_j^2(G)-3)r - 2\lambda_j^2(G) - \lambda_j(G)](p+q) + \lambda_j(G)r\\ &\quad + \lambda_j^2(G)(p-q)^2 - r^3 - 4r^2,\\ c_j &= 2r^2(p+q+1)(p+q+r) - 2(p+q)r\lambda_j^2(G) - 2(p-q)^2\lambda_j^2(G) - r(r+\lambda_j(G))(r+p+q).\end{aligned}$$

Proof It is well known that[3] the $Q(K_{p,q})$-coronal of $K_{p,q}$ is

$$\Gamma_{Q(K_{p,q})}(x) = \frac{(p+q)x-(p-q)^2}{x^2-(p+q)x}.$$

Note that the Q-spectrum of $K_{p,q}$ is $\sigma(Q(K_{p,q})) = (0, p^{(q-1)}, q^{(p-1)}, p+q)$, where $p^{(q-1)}$ denotes the eigenvalue p with multiplicity $q-1$. Thus, by Theorem 1.2, we obtain the required result. □

Theorem 1.2 can enable us to construct infinite pairs of Q-cospectral graphs.

Corollary 1.6 (i) If G_1 and G_2 are two Q-cospectral r-regular graphs, and H is an arbitrary graph, then $G_1 \overline{\star} H$ and $G_2 \overline{\star} H$ are Q-cospectral;

(ii) If G is a regular graph, H_1 and H_2 are two Q-cospectral graphs with $\Gamma_{Q(H_1)}(x) = \Gamma_{Q(H_2)}(x)$, then $G \overline{\star} H_1$ and $G \overline{\star} H_2$ are Q-cospectral.

1.3 L-spectra of $G_1 \overline{\star} G_2$

Theorem 1.3 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an arbitrary graph on n_2 vertices. Then

$$\begin{aligned}\phi(L(G_1 \overline{\star} G_2); x) = (x-2)^{m_1-n_1} \prod_{i=1}^{n_2}(x - r_1 - \mu_i(G_2))^{n_1} \prod_{i=1}^{n_1}[x^2 - (r_1(n_2+1)\\ + \Gamma_{L(G_2)}(x-r_1)\lambda_i^2(G_1) + 2)x + (2n_2+1)r_1\\ + 2\Gamma_{L(G_2)}(x-r_1)\lambda_i^2(G_1) - \lambda_i(G_1)].\end{aligned}$$

Proof With respect to the partition $V(G_1) \cup I(G_1) \cup [\bigcup_{i=1}^{n_1} V^i(G_2)]$ of $V(G_1 \overline{\star} G_2)$, we can write the Laplacian matrix of $G_1 \overline{\star} G_2$ as

$$L(G_1 \overline{\star} G_2) = \begin{pmatrix} (n_2+1)r_1 I_{n_1} & -R & -1_{n_2}^{\mathrm{T}} \otimes A(G_1) \\ -R^{\mathrm{T}} & 2I_{m_1} & 0 \\ -1_{n_2} \otimes A(G_1) & 0 & (L(G_2)+r_1 I_{n_2}) \otimes I_{n_1} \end{pmatrix}.$$

Then the L-characteristic polynomial of $G_1 \overline{\star} G_2$ is given by

$$\begin{aligned}\phi(L(G_1 \overline{\star} G_2)) &= \det \begin{pmatrix} (x-(n_2+1)r_1)I_{n_1} & R & 1_{n_2}^{\mathrm{T}} \otimes A(G_1) \\ R^{\mathrm{T}} & (x-2)I_{m_1} & 0 \\ 1_{n_2} \otimes A(G_1) & 0 & ((x-r_1)I_{n_2} - L(G_2)) \otimes I_{n_1} \end{pmatrix}\\ &= \det(((x-r_1)I_{n_2} - L(G_2)) \otimes I_{n_1}) \cdot \det(S)\\ &= \prod_{i=1}^{n_2}(x - r_1 - \mu_i(G_2))^{n_1} \cdot \det(S),\end{aligned}$$

where

$$S=\begin{pmatrix} (x-(n_2+1)r_1)I_{n_1}-\Gamma_{L(G_2)}(x-r_1)\cdot A^2(G_1) & R \\ R^{\mathrm{T}} & (x-2)I_{m_1} \end{pmatrix}$$

is the Schur complement[16] of $((x-r_1)I_{n_2}-L(G_2))\otimes I_{n_1}$ and

$$\begin{aligned}\det(S)&=(x-2)^{m_1}\det\left[(x-(n_2+1)r_1)I_{n_1}-\Gamma_{L(G_2)}(x-r_1)A^2(G_1)-\frac{1}{x-2}RR^{\mathrm{T}}\right]\\&=(x-2)^{m_1-n_1}\prod_{i=1}^{n_1}[x^2-(r_1(n_2+1)+\Gamma_{L(G_2)}(x-r_1)\lambda_i^2(G_1)+2)x+(2n_2+1)r_1\\&\quad+2\Gamma_{L(G_2)}(x-r_1)\lambda_i^2(G_1)-\lambda_i(G_1)].\end{aligned}$$

Hence, the result follows. □

Corollary 1.7 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be any graph on n_2 vertices. Then the L-spectrum of $G_1\overline{\star}G_2$ consists of

(i) $r_1+\mu_i(G_2)$, repeated n_1 times for each $i=2,3,\cdots,n_2$;

(ii) 2, repeated m_1-n_1 times;

(iii) three roots of the equation, for each $j=1,2,\cdots,n_1$,

$$x^3-(n_2r_1+2r_1+2)x^2+a_jx+b_j=0,$$

where

$$\begin{aligned}a_j&=(n_2+1)r_1^2+(2n_2+3)r_1-n_2\lambda_j^2(G_1)-\lambda_j(G_1),\\b_j&=-(2n_2+1)r_1^2+2n_2\lambda_j^2(G_1)+\lambda_j(G_1)r_1.\end{aligned}$$

Proof Since each row sum of $L(G_2)$ equals 0, then

$$\Gamma_{L(G_2)}(x-r_1)=\frac{n_2}{x-r_1}.$$

The only pole of $\Gamma_{L(G_2)}(x-r_1)$ is $x=r_1$. Note that $\mu_1(G_2)=0$. Thus, by Theorem 1.3, $r_1+\mu_i(G_2)$ is an L-eigenvalue of $G_1\overline{\star}G_2$, repeated n_1 times for each $i=2,3,\cdots,n_2$, and 2 is also an L-eigenvalue of $G_1\overline{\star}G_2$, repeated m_1-n_1 times. The remaining L-eigenvalues are obtained by solving the equation as above. □

As an application of Theorem 1.3, we may construct infinite pairs of L-cospectral graphs.

Corollary 1.8 (i) If G_1 and G_2 are two L-cospectral r-regular graphs, and H is an arbitrary graph, then $G_1\overline{\star}H$ and $G_2\overline{\star}H$ are L-cospectral.

(ii) If G is a regular graph, H_1 and H_2 are two L-cospectral graphs, then $G\overline{\star}H_1$ and $G\overline{\star}H_2$ are L-cospectral.

Let G be a connected graph with n vertices and L-eigenvalues $0=\mu_1(G)\le\mu_2(G)\le\cdots\le\mu_n(G)$. It is well known that[4] the number of spanning trees of G is

$$t(G)=\frac{\mu_2(G)\mu_3(G)\cdots\mu_n(G)}{n}.$$

Thus, by Corollary 1.7, we can obtain the following result.

Corollary 1.9 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an arbitrary graph on n_2 vertices. Then the number of spanning trees of $G_1 \barwedge G_2$ is

$$t(G_1 \barwedge G_2) = \frac{2^{m_1-n_1}(r_1^2+(2n_2+2)r_1)\prod_{i=2}^{n_2}(r_1+\mu_i(G_2))^{n_1}\prod_{i=2}^{n_1}(2n_2r_1^2+r_1^2-2n_2\lambda_i^2(G_1)-\lambda_i(G_1)r_1)}{n_1+m_1+n_1n_2}.$$

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局部剖分邻接冠图的谱

朱雪琴 [1], 田贵贤 [1], 崔淑玉 [2]

(1. 浙江师范大学数理与信息工程学院, 金华, 浙江, 321004; 2. 浙江师范大学行知学院, 金华, 浙江, 321004)

摘要: 设 G_1, G_2 是两个简单连通图, 图 G_1, G_2 的局部剖分邻接冠图 $G_1 \barwedge G_2$ 是指复制一个 G_1 和 $|V(G_1)|$ 个 G_2, 图 G_1 的第 i 个点的邻点与复制的第 i 个图 G_2 的每一个点相连接, 然后在 G_1 每一条边上插入一个新的点而得到的图类. 本文利用两个图 G_1, G_2 的邻接谱、Laplacian 谱和无符号 Laplacian 谱刻画了局部剖分邻接冠图 $G_1 \barwedge G_2$ 的邻接谱、Laplacian 谱和无符号 Laplacian 谱. 另外, 本文利用上述结果构造出了若干对邻接同谱图、Laplacian 同谱图和无符号 Laplacian 同谱图. 进一步地, 本文也利用两个因子图 G_1, G_2 的 Laplacian 谱计算出了局部剖分邻接冠图 $G_1 \barwedge G_2$ 的生成树数目.

关键词: 邻接矩阵; Laplacian 矩阵; 无符号 Laplacian 矩阵; 图的谱; 局部剖分邻接冠图