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Relatively Gorenstein-projective Modules

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Abstract: Let A be an extension ring of a ring B, that is, B is a subring of A with the same identity. We denote by $\mathcal{P}(A,B)$ the category of all the relatively projective modules. For this extension $B \hookrightarrow A$, we introduce relatively Gorenstein-projective modules. As Gorenstein-projective modules are closely related to projective modules and there are some good results about Gorenstein dimensions, we want to give a similar relationship between relatively Gorenstein-projective modules and relatively projective modules.

The main results are: (1) Let $B \hookrightarrow A$ be an extension of rings with the same identity. Then the category of all the relatively Gorenstein projective modules is relatively resolving. (2) Let $B \hookrightarrow A$ be an extension of rings with the same identity. If $\mathrm{gl.dim}(A,B) \le n$, then every relatively Gorenstein-projective module is relatively projective, where $\mathrm{gl.dim}(A,B)$ represents the supreme of relatively projective dimension of all the A-modules.

Keywords: relative Gorenstein projective modules; relative global dimension; relatively resolving

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0 Introduction

Let A be an Artin algebra. In the representation theory of Artin algebra, there is a famous finitistic dimension conjecture (see [1–2]), which says that there exists a uniform bound for the finite projective dimensions of all finitely generated (left) A-modules of finite projective dimension. It is over 50 years old and the conjecture has not been completely resolved. The conjecture is very important because it is not an isolated one but closely related to some other homological conjectures (see [12] for details).

Hochschild[8] in 1956 introduced relatively projective modules and relatively injective modules, and then the relative Tor and relative Ext functors, which we will always use in this paper. For more relatively homological properties, one can refer to [7, 10]. Xi and Xu[13] in 2013 applied relatively projective modules to the conjecture of finitistic dimension and proved the following: Let A be an Artin algebra and B be a subalgebra of A such that the radical of B is a left ideal in A. If the category of all finitely generated relatively projective A-modules is closed under taking A-syzygies, then the conjecture of finitistic dimension is true. This interests us to study the category of all relatively projective modules and two theorems below[14] give us an idea about how to study it.

Proposition 0.1 Suppose that A is an Artin algebra and e is an idempotent element of A with $\mathrm{proj.dim}(Ae_{eAe}) < \infty$. If $\mathrm{fin.dim}(A) < \infty$, then $\mathrm{fin.dim}(eAe) < \infty$.

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Theorem 0.1 Suppose that A is an Artin algebra and e is an idempotent element of A with G-dim$(Ae_{eAe}) < \infty$. If fin.dim$(A) < \infty$, then fin.dim$(eAe) < \infty$.

Comparing the two results, we can find that Gorenstein projective modules are closely related to projective modules and have nice homological properties[9].

In order to consider the finitistic dimension conjecture and study the category of all relatively projective modules, affected by the method above, we consider the general ring extensions and give a new definition called relatively Gorenstein-projective module. We will prove that the relatively Gorenstein-projective modules are closely related to relatively projective modules and have nice homological properties. Our main results are as follows:

Theorem 0.2[14] Let $B \hookrightarrow A$ be an extension of rings with the same identity. Then the category of all relatively Gorenstein projective modules is relatively resolving, in the sense that if $0 \to X' \to X \to X'' \to 0$ is an (A, B)-exact sequence in A-Mod, where X'' is relatively Gorenstein projective, then X is relatively Gorenstein projective if and only if X' is relatively Gorenstein projective.

Theorem 0.3[14] If gl.dim$(A, B) \leq n < \infty$, then every relatively Gorenstein projective module is relatively projective, that is, $\mathcal{GP}(A, B) = \mathcal{P}(A, B)$.

The paper is organized as follows: In Section 1, we recall some definitions and basic facts. The new definitions and properties are given in Section 2. In Section 3, we present some examples to show the significance of this paper.

1 Preliminaries

In this section, we recall some definitions and basic results that will be used in the paper.

Throughout this paper, we assume that A is an extension ring of B, that is, B is a subring of A with the same identity. A-Mod represents the category of all left A-modules, relatively projective module will always be denoted by (A, B)-projective module, the category of all relatively projective modules is denoted by $\mathcal{P}(A, B)$, $\mathcal{I}(A, B)$ represents the category of all relatively injective modules, $\mathcal{P}(A)$ represents the category of all projective A-modules, and $\mathcal{GP}(A)$ represents the category of all Gorenstein projective A-modules. They are all the full subcategories of A-Mod.

Suppose that $\mathcal{X} \subseteq A$-Mod is a subcategory of A-Mod. The long exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in A-Mod is called $\mathrm{Hom}_A(\mathcal{X}, -)$ exact, if applying $\mathrm{Hom}_A(X, -)$ to the sequence above is still exact for any $X \in \mathcal{X}$. Dually, $\mathrm{Hom}_A(-, \mathcal{X})$ exact has the similar meaning.

Definition 1.1[8] Suppose that A is an extension ring of B. A short exact sequence in A-Mod

$$0 \to U \to V \to W \to 0$$

is called (A, B)-exact, if it splits as B-modules. A long exact sequence in A-Mod is called (A, B)-exact, if it is (A, B)-exact when splitting into short exact sequences.

Next, we give an equivalent description of (A, B)-exact sequences.

Lemma 1.1 A short exact sequence in A-Mod is (A, B)-exact if and only if the exact sequence is $\mathrm{Hom}_A(\mathcal{P}(A, B), -)$ exact.

Let us recall the definitions of relatively projective module and relatively injective module. For the details, the readers can see [8].

Definition 1.2 An A-module M is called relatively projective (or (A,B)-projective), if for any (A,B)-exact sequence $0 \to U \xrightarrow{p} V \xrightarrow{q} W \to 0$ and a homomorphism $g : M \to W$ of A-modules, there exists a homomorphism $g' : M \to V$ of A-modules such that $g'q = g$. Dually, one can define the relatively injective module.

Remark 1.1 1) M is (A,B)-projective if and only if M is a summand of $A \otimes_B M$. For any $Y \in B$-Mod, $A \otimes_B Y$ is (A,B)-projective.

2) For any $X \in A$-Mod, $0 \to K \to A \otimes_B X \to X \to 0$ is (A,B)-exact, where $A \otimes_B X \to X$ is the multiplication map and K is the kernel of $A \otimes_B X \to X$.

Definition 1.3[13] For an A-module X, the relative projective dimension is defined to be the minimal number n such that there is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$ satisfying:

(1) All P_j are (A,B)-projective.

(2) $\mathrm{Hom}_A(\mathcal{P}(A,B),-)$ is exact.

If such an exact sequence does not exist, then we say that the relative projective dimension of X is infinite. The relative global dimension represents the supremum of the relative projective dimensions of A-modules, denoted by $\mathrm{gl.dim}(A,B)$.

Also, we have the relative derived functors $\mathrm{Ext}_{(A,B)}$ and $\mathrm{Tor}^{(A,B)}$, see [3, 8] for details. Here, we list some properties of the relatively derived functors which will be used in the next section.

1) For relative projective module P, $\mathrm{Ext}^n_{(A,B)}(P,X) = 0$, for any $X \in A$-Mod, $n \geq 1$.

2) Let $0 \to U \to V \to W \to 0$ be an (A,B)-exact sequence, for any $X \in A$-Mod, we can get two long exact sequences below:

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(X,U) \to \mathrm{Hom}_A(X,V) \to \mathrm{Hom}_A(X,W) \to \mathrm{Ext}^1_{(A,B)}(X,U) \\ &\to \mathrm{Ext}^1_{(A,B)}(X,V) \to \mathrm{Ext}^1_{(A,B)}(X,W) \to \mathrm{Ext}^2_{(A,B)}(X,U) \to \cdots \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(W,X) \to \mathrm{Hom}_A(V,X) \to \mathrm{Hom}_A(U,X) \to \mathrm{Ext}^1_{(A,B)}(W,X) \\ &\to \mathrm{Ext}^1_{(A,B)}(V,X) \to \mathrm{Ext}^1_{(A,B)}(U,X) \to \mathrm{Ext}^2_{(A,B)}(W,X) \to \cdots . \end{aligned}$$

3) If $0 \to U \to V \to W \to 0$ is an (A,B)-exact sequence with $W \in \mathcal{P}(A,B)$, then it splits as A-modules, that is, $V \cong U \oplus W$ as A-modules.

4) Applying $\mathrm{Hom}_A(\mathcal{P}(A,B),-)$ to an arbitrary (A,B)-exact sequence is still exact.

Next, we recall the definition of Gorenstein projective module and some basic properties, see [4–6] for details.

Definition 1.4[6] Let R be an associative ring with identity. $G \in R$-Mod is called Gorenstein projective if there exists an exact sequence in R-Mod

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with the following properties:

1) All P_i and P^i are projective;

2) $\mathrm{Hom}_R(-,\mathcal{P}(R))$ is exact;

3) $G \cong \mathrm{Im}(P_0 \to P^0)$.

Remark 1.2 Every projective R-module is Gorenstein projective, that is, $\mathcal{P}(R) \subset \mathcal{GP}(R)$. All the images, kernel and cokernel of $\mathbf{P}$ are Gorenstein projective modules.

There is a main theorem about the category of all Gorenstein projective modules.

Theorem 1.1[8] The class of all Gorenstein projective modules is resolving, in the sense that if $0 \to X' \to X \to X'' \to 0$ is a short exact sequence in R-Mod, where $X'' \in \mathcal{GP}(R)$, then X' is Gorenstein projective if and only if X is Gorenstein projective.

2 Relative Gorenstein Projective (Injective) Modules

In this section, we shall give the new definitions and the main theorems.

Definition 2.1 Let A be an extension ring of a ring B. An A-module X is called relatively Gorenstein projective (or (A, B)-Gorenstein projective) if there exists a $\mathrm{Hom}_A(\mathcal{P}(A, B), -)$ exact and $\mathrm{Hom}_A(-, \mathcal{P}(A, B))$ exact sequence in A-Mod

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \cdots$$

such that $X \cong \mathrm{Im}(P_0 \to \mathcal{P}^0)$, where $P_i, P^i \in \mathcal{P}(A, B)$ for all i.

The category of all relatively Gorenstein projective modules is denoted by $\mathcal{GP}(A, B)$.

Remark 2.1 By Lemma 1.1, we know that the condition $\mathrm{Hom}_A(\mathcal{P}(A, B), -)$ exact is equivalent to $\mathbf{P}$ (A, B)-exact.

2) Every (A, B)-projective module is (A, B)-Gorenstein projective. In particular, every projective A-module is (A, B)-Gorenstein projective.

3) All the images, kernels and cokernels of $\mathbf{P}$ are in $\mathcal{GP}(A, B)$.

4) If B is semisimple, in the sense that B is semisimple as a B-module, then every (A, B)-projective module is projective and every (A, B)-Gorenstein projective module is Gorenstein projective, that is, $\mathcal{P}(A, B)$=$\mathcal{P}(A)$, $\mathcal{GP}(A, B)$=$\mathcal{GP}(A)$. For this reason, the module in $\mathcal{GP}(A, B)$ is called relatively Gorenstein projective.

In fact, the categories we inferred have the following relations:

$$\begin{array}{ccc} \mathcal{P}(A) & \hookrightarrow & \mathcal{P}(A,B) \\ \downarrow & & \downarrow \\ \mathcal{GP}(A) & & \mathcal{GP}(A,B) \end{array}$$

Definition 2.2 Suppose that $\mathcal{X} \subseteq A$-Mod is a full subcategory of A-Mod. $\mathcal{X}$ is called relatively resolving if $\mathcal{P}(A) \subseteq \mathcal{X}$, and for every short (A, B)-exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$, the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

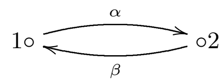
We get our main theorems.

Theorem 2.1 Let $B \hookrightarrow A$ be an extension of rings with the same identity. Then the category of all relatively Gorenstein projective modules is relatively resolving, in the sense that if $0 \to X' \to X \to X'' \to 0$ is an (A, B)-exact sequence in A-Mod, where X'' is relatively Gorenstein projective, then X is relatively Gorenstein projective if and only if X' is relatively Gorenstein projective.

Theorem 2.2 If $\mathrm{gl.dim}(A, B) \leq n < \infty$, then every relatively Gorenstein projective module is relatively projective, that is, $\mathcal{GP}(A, B) = \mathcal{P}(A, B)$.

We give an example to show that there may exist relatively Gorenstein-projective module which is not relatively projective if gl.dim(A, B) is infinite.

Example 2.1 Let A be the path algebra given by the quiver



with relations: $\alpha\beta = \beta\alpha = 0$. B is the subalgebra generated by the primitive idempotent elements of A corresponding to the vertices of the quiver. Then B is a semi-simple algebra and every relatively projective module is projective, that is, $\mathcal{P}(A, B) = \mathcal{P}(A)$. There exists an exact sequence in A-module

$$\cdots \xrightarrow{f} \begin{matrix}1\\2\end{matrix} \xrightarrow{g} \begin{matrix}2\\1\end{matrix} \xrightarrow{f} \begin{matrix}1\\2\end{matrix} \xrightarrow{g} \begin{matrix}2\\1\end{matrix} \rightarrow \cdots$$

such that the simple module $S(1) \cong \operatorname{Im} g$, so $S(1)$ is relatively Gorenstein projective, $S(1) \in \mathcal{GP}(A, B) = \mathcal{GP}(A)$, but $S(1) \notin \mathcal{P}(A, B) = \mathcal{P}(A)$.

Now, we prove the main theorems. First, we give some definitions and lemmas relevant to the proof.

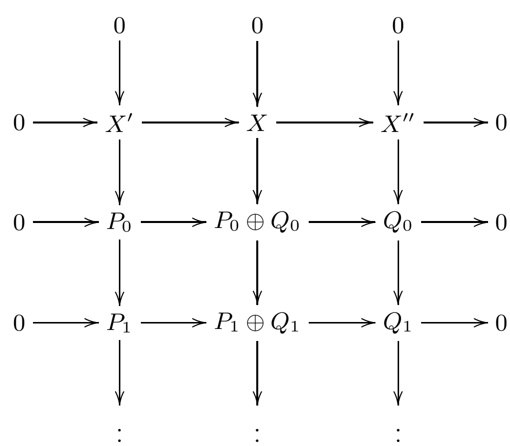
Definition 2.3 A right $\mathcal{P}(A, B)$-resolution of an A-module M is that there exists an (A, B)-exact sequence $\mathbf{P}^+ = 0 \to M \to P^0 \to P^1 \to \cdots$ with $P^i \in \mathcal{P}(A, B)$ for all $i \geq 0$. Moreover, we say that $\mathbf{P}^+$ is co-proper if $\operatorname{Hom}_A(\mathbf{P}^+, Q)$ is exact for all $Q \in \mathcal{P}(A, B)$.

${}^{\perp}\mathcal{P}(A, B) := \{X \in A\text{-Mod} \mid \operatorname{Ext}^i_{(A,B)}(X, \mathcal{P}(A, B)) = 0, \text{ for all } i > 0\}$.

Similar with [9, Prop. 2.3], we have the following lemma in order to describe the relatively Gorenstein projective module.

Lemma 2.1 An A-module M is (A, B)-Gorenstein projective if and only if $M \in {}^{\perp}\mathcal{P}(A, B)$ and admits a co-proper right $\mathcal{P}(A, B)$-resolution.

Lemma 2.2 (Horseshoe Lemma) Let $0 \to X' \to X \to X'' \to 0$ be a short (A, B)-exact sequence with X', $X'' \in \mathcal{GP}(A, B)$. Suppose that $0 \to X' \to P_0 \to P_1 \to \cdots$ is a co-proper right $\mathcal{P}(A, B)$-resolution of X', and $0 \to X'' \to Q_0 \to Q_1 \to \cdots$ is a co-proper right $\mathcal{P}(A, B)$-resolution of X''. Then we can construct the following commutative diagram such that the middle column is a co-proper right $\mathcal{P}(A, B)$-resolution of X.



Proof For $P_0 \in \mathcal{P}(A,B)$, we have a long exact sequence

$$0 \to \mathrm{Hom}_A(X'', P_0) \to \mathrm{Hom}_A(X, P_0) \to \mathrm{Hom}_A(X', P_0) \to \mathrm{Ext}^1_{(A,B)}(X'', P_0) \to \cdots$$

and $\mathrm{Ext}^1_{(A,B)}(X'', P_0) = 0$ because $X'' \in \mathcal{P}(A,B)$, then we have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow & \swarrow & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0' & \longrightarrow & K & \longrightarrow & K_0'' & \longrightarrow & 0
\end{array}$$

where K_0', K, and K_0'' are cokernels of each corresponding homomorphism. By Snake Lemma, $0 \to K_0' \to K \to K_0'' \to 0$ is exact.

For any $Q \in \mathcal{P}(A,B)$, applying $\mathrm{Hom}_A(Q,-)$ to the commutative diagram above and by 3×3 [11, Lemma, p. 96], we can get the following communicative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_A(Q,X') & \longrightarrow & \mathrm{Hom}_A(Q,X) & \longrightarrow & \mathrm{Hom}_A(Q,X'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_A(Q,P_0) & \longrightarrow & \mathrm{Hom}_A(Q,P_0 \oplus Q_0) & \longrightarrow & \mathrm{Hom}_A(Q,Q_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_A(Q,K_0') & \longrightarrow & \mathrm{Hom}_A(Q,K) & \longrightarrow & \mathrm{Hom}_A(Q,K_0'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

So $0 \to X \to P_0 \oplus Q_0 \to K \to 0$ and $0 \to K_0' \to K \to K_0'' \to 0$ are both (A,B)-exact. Now repeating the argument by using (A,B)-exact sequence $0 \to K_0' \to K \to K_0'' \to 0$, we have the result. □

Lemma 2.3 Let $f : M \to N$ be a homomorphism of A-modules and consider the following diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & P^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f} & & & & & & & & \\
0 & \longrightarrow & N & \longrightarrow & Q^0 & \longrightarrow & Q^1 & \longrightarrow & Q^2 & \longrightarrow & \cdots
\end{array}$$

where the upper row is a co-proper right $\mathcal{P}(A,B)$-resolution of M and the lower row is a right

$\mathcal{P}(A,B)$-resolution of N. Then $f : M \to N$ induces a chain map of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & P^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\
0 & \longrightarrow & Q^0 & \longrightarrow & Q^1 & \longrightarrow & Q^2 & \longrightarrow & \cdots
\end{array}$$

with the property that the square

$$\begin{array}{ccccc}
0 & \longrightarrow & M & \longrightarrow & P^0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} \\
0 & \longrightarrow & N & \longrightarrow & Q^0
\end{array}$$

commutes.

Furthermore, the chain map above is uniquely determined up to homotopy by this property.

Proof Similar to [9, Prop. 1.8] and use the fact that for any $P \in \mathcal{GP}(A,B)$, $P \in {}^{\perp}\mathcal{P}(A,B)$, then Lemma 2.3 holds. □

Now, we can prove Theorem 2.1.

Let $0 \to X' \to X \to X'' \to 0$ be an (A,B)-exact sequence with $X'' \in \mathcal{GP}(A,B)$. To prove that $\mathcal{GP}(A,B)$ is relatively resolving, we first suppose $X' \in \mathcal{GP}(A,B)$. For any $Q \in \mathcal{P}(A,B)$, we can get a long exact sequence

$$\begin{aligned}
0 &\to \mathrm{Hom}_A(X'',Q) \to \mathrm{Hom}_A(X,Q) \to \mathrm{Hom}_A(X',Q) \to \mathrm{Ext}^1_{(A,B)}(X'',Q) \\
&\to \mathrm{Ext}^1_{(A,B)}(X,Q) \to \mathrm{Ext}^1_{(A,B)}(X',Q) \to \cdots .
\end{aligned}$$

By Lemma 2.1, we know that X' and X'' are in ${}^{\perp}\mathcal{P}(A,B) \Rightarrow X \in {}^{\perp}\mathcal{P}(A,B)$. Also by Lemma 2.1, both X' and X'' have a co-proper right $\mathcal{P}(A,B)$-resolution, then by Lemma 2.2 (Horseshoe Lemma), so does X. By Lemma 2.1, we finally get $X \in \mathcal{P}(A,B)$.

Next, we assume $X \in \mathcal{P}(A,B)$. Then $X \in {}^{\perp}\mathcal{P}(A,B)$, and by the long exact sequence above, we can get $X' \in {}^{\perp}\mathcal{P}(A,B)$. Our aim is to give X' a co-proper right $\mathcal{P}(A,B)$-resolution. Suppose that $\mathbf{X} = 0 \to X \to P^0 \to P^1 \to \cdots$ is a co-proper right $\mathcal{P}(A,B)$-resolution of X. $\mathbf{X}_1 = 0 \to X'' \to Q^0 \to Q^1 \to \cdots$ is a co-proper right $\mathcal{P}(A,B)$-resolution of X''. Lemma 2.3 gives a chain map of $\mathbf{X} \to \mathbf{X}_1$, which lifts the homomorphism of $X \to X'$. Let $\mathbf{C}$ denote the mapping cone of $\mathbf{X} \to \mathbf{X}_1$. Since $\mathbf{X} \to \mathbf{X}_1$ is quasi-isomorphism, $\mathbf{C}$ is exact. Actually,

$$\mathbf{C} = 0 \to X \to P^0 \oplus X'' \to P^1 \oplus Q^0 \to \cdots .$$

For any $Q \in \mathcal{P}(A,B)$, the complex

$$\mathrm{Hom}_A(\mathbf{C},Q) = \cdots \to \mathrm{Hom}_A(P^1 \oplus Q^0, Q) \to \mathrm{Hom}_A(P^0 \oplus X'', Q) \to \mathrm{Hom}_A(X,Q) \to 0$$

is just isomorphic to the mapping cone of the quasi-isomorphism of $\mathrm{Hom}_A(\mathbf{X}_1,Q) \to \mathrm{Hom}_A(\mathbf{X},Q)$, so $\mathrm{Hom}_A(\mathbf{C}, Q)$ is exact. We have the commutative diagram below:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^1 \oplus Q^0 & \longrightarrow & P^1 \oplus Q^0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^0 & \longrightarrow & P^0 \oplus X'' & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbf{X}_0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & 0
\end{array}
$$

□

Claim: $\mathbf{X}_0$ is a co-proper right $\mathcal{P}(A, B)$-resolution of X'.

Since $\mathbf{C}$ and $\mathbf{D}$ are exact complexes, $\mathbf{X}_0$ is an exact complex in A-Mod.

$$\mathbf{X}_0 = 0 \to X' \to P^0 \to P^1 \oplus Q^0 \to \cdots .$$

Applying $\mathrm{Hom}_A(Q, -)$ to $\mathbf{X}_0$, we get a complex

$$0 \to \mathrm{Hom}_A(Q, X') \to \mathrm{Hom}_A(Q, P^0) \to \mathrm{Hom}_A(Q, P^1 \oplus Q^0) \to \cdots .$$

We have known that $0 \to X \to P^0 \to P^1 \to \cdots$ and $0 \to X'' \to Q^0 \to Q^1 \to \cdots$ are (A, B)-exact, so $0 \to \mathrm{Hom}_A(Q, X) \to \mathrm{Hom}_A(Q, P^0) \to \mathrm{Hom}_A(Q, P^1) \to \cdots$ and $0 \to \mathrm{Hom}_A(Q, X'') \to \mathrm{Hom}_A(Q, Q^0) \to \mathrm{Hom}_A(Q, Q^1) \to \cdots$ are exact, then the mapping cone $0 \to \mathrm{Hom}_A(Q, X) \to \mathrm{Hom}_A(Q, P^0) \oplus \mathrm{Hom}_A(Q, X'') \to \mathrm{Hom}_A(Q, P^1) \oplus \mathrm{Hom}_A(Q, Q^0) \to \cdots$ is exact, i.e., $0 \to \mathrm{Hom}_A(Q, X) \to \mathrm{Hom}_A(Q, P^0) \oplus \mathrm{Hom}_A(Q, X'') \to \mathrm{Hom}_A(Q, P^1 \oplus Q^0) \to \mathrm{Hom}_A(Q, P^2 \oplus Q^1) \to \cdots$ is exact. Let $K = \mathrm{coker}(\mathrm{Hom}_A(Q, X) \to \mathrm{Hom}_A(Q, P^0) \oplus \mathrm{Hom}_A(Q, X''))$. Because $0 \to X' \to X \to X'' \to 0$ is an (A, B)-exact sequence, we have the commutative diagram below:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Hom}_A(Q, X') & \longrightarrow & \mathrm{Hom}_A(Q, X) & \longrightarrow & \mathrm{Hom}_A(Q, X'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}_A(Q, P_0) & \longrightarrow & \mathrm{Hom}_A(Q, P_0) \oplus \mathrm{Hom}_A(Q, X'') & \longrightarrow & \mathrm{Hom}_A(Q, X'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

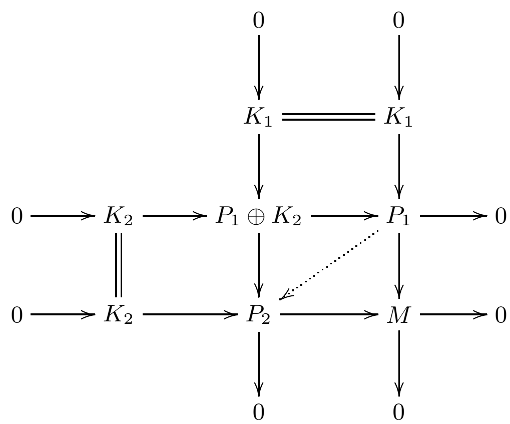
Then we can get that $0 \to \mathrm{Hom}_A(Q, X') \to \mathrm{Hom}_A(Q, P^0) \to K \to 0$ is exact, so $0 \to \mathrm{Hom}_A(Q, X') \to \mathrm{Hom}_A(Q, P^0) \to \mathrm{Hom}_A(Q, P^1 \oplus Q^0) \to \mathrm{Hom}_A(Q, P^2 \oplus Q^1) \to \cdots$ is exact, i.e., $\mathbf{X}_0$ is (A, B)-exact. Now we know that $\mathbf{X}_0$ is a right $\mathcal{P}(A, B)$-resolution of X'. To see that it is co-proper, for all $Q \in \mathcal{P}(A, B)$.

$$0 \to \mathrm{Hom}_A(\mathbf{D}, Q) \to \mathrm{Hom}_A(\mathbf{C}, Q) \to \mathrm{Hom}_A(\mathbf{X}_0, Q) \to 0$$

is an exact sequence of complexes. Because $\mathrm{Hom}_A(\mathbf{D}, Q)$ and $\mathrm{Hom}_A(\mathbf{C}, Q)$ are exact, so is $\mathrm{Hom}_A(\mathbf{X}_0, Q)$. Finally we have that $\mathbf{X}_0$ is a co-proper right $\mathcal{P}(A, B)$-resolution of X', so $X' \in \mathcal{GP}(A, B)$. $\mathcal{GP}(A, B)$ is relatively resolving.

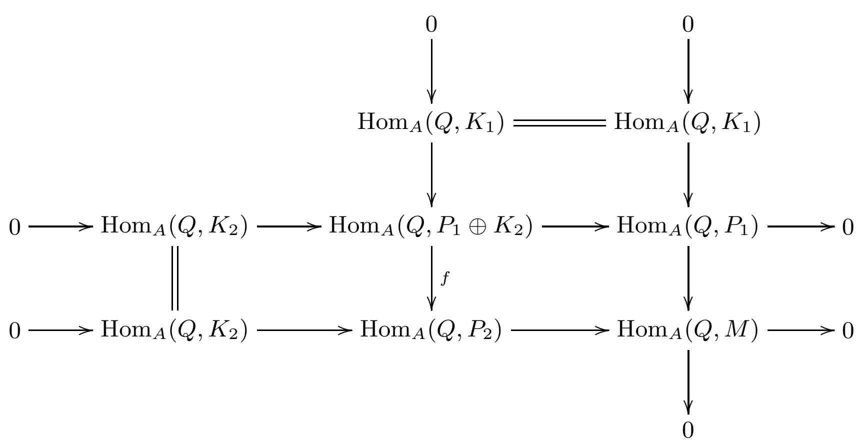
Corollary 2.1 (Schanuel's Lemma) Let $0 \to K_1 \to P_1 \to M \to 0$ and $0 \to K_2 \to P_2 \to M \to 0$ be two (A, B)-exact sequences in A-Mod, with $P_1 \in \mathcal{P}(A, B)$, $P_2 \in \mathcal{GP}(A, B)$. Then $K_1 \in \mathcal{GP}(A, B)$ if and only if $K_2 \in \mathcal{GP}(A, B)$.

Proof From the assumption, we can get the commutative diagram below:



So $0 \to K_1 \to P_1 \oplus K_2 \to P_2 \to 0$ is exact.

Moreover, applying $\mathrm{Hom}_A(Q, -)$ to the commutative above with $Q \in \mathcal{P}(A, B)$, we can get the following commutative diagram:



By Snake Lemma, f is epic, so $0 \to K_1 \to P_1 \oplus K_2 \to P_2 \to 0$ is (A, B)-exact. By Theorem 2.1,

$\mathcal{GP}(A,B)$ is relatively resolving and closed under direct sums and summands, so $K_1 \in \mathcal{GP}(A,B)$ if and only if $K_2 \in \mathcal{GP}(A,B)$. □

Further, we give the general statement without proof.

Corollary 2.2 Let $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ and $0 \to \widetilde{K}_n \to \widetilde{P}_{n-1} \to \cdots \to \widetilde{P}_0 \to M \to 0$ be two (A,B)-exact sequences in A-Mod with $P_i, \widetilde{P}_i$ (A,B)-projective. Then $K_n \in \mathcal{GP}(A,B)$ if and only if $\widetilde{K}_n \in \mathcal{GP}(A,B)$.

The next aim is to prove Theorem 2.2. First, we have the following lemma.

Lemma 2.4 If $\mathrm{gl.dim}(A,B) = 1$, then $X \in \mathcal{GP}(A,B)$ if and only if $X \in \mathcal{P}(A,B)$, that is, $\mathcal{GP}(A,B) = \mathcal{P}(A,B)$.

Proof "$\Leftarrow$" Trivial.

"$\Rightarrow$" Because $\mathrm{gl.dim}(A,B) = 1$, suppose that $0 \to P_1 \to P_0 \to X \to 0$ is a short (A,B)-exact sequence with $P_0, P_1 \in \mathcal{P}(A,B)$. By the assumption $X \in \mathcal{GP}(A,B)$, $X \in {}^{\perp}\mathcal{P}(A,B)$ by Lemma 2.1, and we have a long exact sequence

$$0 \to \mathrm{Hom}_A(X, P_1) \to \mathrm{Hom}_A(X, P_0) \to \mathrm{Hom}_A(X, X) \to \mathrm{Ext}^1_{(A,B)}(X, P_1) \to \cdots,$$

so the short exact sequence splits as A-modules, i.e., $P_0 \cong P_1 \oplus X$. From the definition of relatively projective module, we can easily know that $\mathcal{P}(A,B)$ is closed under direct sums and summands, so $X \in \mathcal{P}(A,B)$. This completes the proof of the lemma. □

Proof of Theorem 2.2 Suppose $\mathrm{gl.dim}(A,B) \leq n < \infty$. For any $X \in \mathcal{GP}(A,B)$, there exists an (A,B)-exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

with $P_i \in \mathcal{P}(A,B)$ for $0 \leq i \leq n$.

Let $K_i = \mathrm{Ker}(P_i \to P_{i-1})$ for $0 \leq i \leq n-1$. Because $0 \to K_0 \to P_0 \to X \to 0$ is (A,B)-exact and $X \in \mathcal{GP}(A,B)$, by Theorem 2.1, we can get that K_0 is in $\mathcal{GP}(A,B)$. Inductively, we know that all K_i are in $\mathcal{GP}(A,B)$. In particular $K_{n-2} \in \mathcal{GP}(A,B)$, and because $0 \to P_n \to P_{n-1} \to K_{n-2} \to 0$ is (A,B)-exact, by the process of proving Lemma 2.4 we have K_{n-2} is relatively projective. Inductively, we can get that X is in $\mathcal{P}(A,B)$. This finishes the proof of Theorem 2.2. □

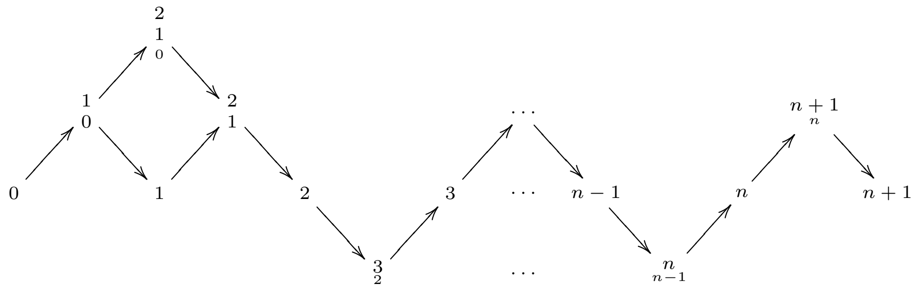
3 Example

In this section, we give an example related to Theorem 2.2.

Let A be the path algebra defined by the quiver

$$n+1\circ \xrightarrow{\alpha_n} \circ\cdots\circ \xrightarrow{\alpha_2} \circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_0} \circ 0$$

with relations: $\alpha_n\alpha_{n-1} = \cdots = \alpha_2\alpha_1 = 0$. Let B be the subalgebra of A generated by α_0 and the primitive idempotent elements of A corresponding to the vertices of the quiver. The Auslander-Reiten quiver of this algebra can be drawn as follows:



The number of non-isomorphic indecomposable modules of A is $2n+4$.

$$0 \qquad 1 \qquad \begin{array}{c}1\\0\end{array} \qquad \begin{array}{c}2\\1\\0\end{array} \qquad \begin{array}{c}3\\2\end{array} \qquad \begin{array}{c}4\\3\end{array} \qquad \cdots \qquad \begin{array}{c}n+1\\n\end{array}$$

are the non-isomorphic indecomposable relatively projective modules, and

$$2 \qquad 3 \qquad \cdots \qquad n+1 \qquad \begin{array}{c}2\\1\end{array}$$

are the non-relatively projective ones. For the simple module $S(n+1)$, there exists an exact sequence

$$0 \to \begin{array}{c}1\\0\end{array} \to \begin{array}{c}2\\1\\0\end{array} \to \begin{array}{c}3\\2\end{array} \to \cdots \to \begin{array}{c}n\\n-1\end{array} \to \begin{array}{c}n+1\\n\end{array} \to n+1 \to 0.$$

One can check that the relatively projective dimensions of the simple modules $2, 3, \cdots, n$ are all less than n. For the indecomposable module ${}^{2}_{1}$, there exists an exact sequence

$$0 \to \begin{array}{c}1\\0\end{array} \to 1 \oplus \begin{array}{c}2\\1\\0\end{array} \to \begin{array}{c}2\\1\end{array} \to 0$$

where $A \otimes_B {}^{2}_{1} \cong 1 \oplus \begin{smallmatrix}2\\1\\0\end{smallmatrix}$, so this short exact sequence is (A,B)-exact. Finally, we get

$$\mathrm{gl.dim}(A,B) = n.$$

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相对 Gorenstein 投射模

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摘要: 设环 A 是环 B 的扩张环, 即 B 是与 A 有相同单位的 A 的子环. 记 $\mathcal{P}(A,B)$ 是由所有相对投射模构成的范畴. 对于扩张 $B \hookrightarrow A$, 本文介绍相对 Gorenstein 投射模的概念. 由于 Gorenstein 投射模与投射模具有紧密的联系, 并且关于 Gorenstein 维数有较好的性质, 本文想给出相对 Gorenstein 投射模和相对投射模之间类似的关系. 本文主要结果是: (1) 设 $B \hookrightarrow A$ 是具有相同单位的环的扩张, 则由所有相对 Gorenstein 投射模构成的范畴是相对可解的. (2) 设 $B \hookrightarrow A$ 是具有相同单位的环的扩张, 若 $\mathrm{gl.dim}(A,B) \leq n$, 则每一个相对 Gorenstein 投射模都是相对投射的, 其中 $\mathrm{gl.dim}(A,B)$ 表示所有 A- 模的相对投射维数的上确界.

关键词: 相对 Gorenstein 投射模; 相对整体维数; 相对可解