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On the Error Term for the Sum of the Coefficients of Dedekind Zeta-function Over Square Numbers

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Abstract: Suppose that E is an algebraic number field over the rational field $\mathbb{Q}$. Let $a(n)$ be the number of integral ideals in E with norm n. Let also $\Delta(x)$ denote the remainder term in the asymptotic formula for the average behavior $\sum_{n\le x}(a(n^2))^l$. In this paper, the sharp bound for

$$\int_1^X \Delta^2(x)\mathrm{d}x$$

is given by analytical method. This result constitutes an improvement upon that of Lü and Yang [*J. Number Theory*, 2011, 131: 1924-1938] for the remainder terms on average.

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1 Introduction and Main Results

Let E be a number field of degree d over the rational field $\mathbb{Q}$, and $\zeta(s,E)$ be its Dedekind zeta-function. Thus

$$\zeta(s,E)=\sum_{\mathfrak{a}}\frac{1}{(N\mathfrak{a})^s}\quad(\mathfrak{Re}(s)>1),$$

where $\mathfrak{a}$ runs over all integral ideals of the field E, and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. If $a(n)$ denotes the number of integral ideals in E with norm n, then we have

$$\zeta(s,E)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s},\quad s=\sigma+\mathrm{i}t,\quad \sigma>1. \tag{1}$$

It is an important problem to study the function $\sum_{n\le x}a^l(n)$. In 1927, Landau[8] first proved that

$$\sum_{n\le x}a(n)=\alpha x+O\big(x^{1-\frac{2}{d+1}+\varepsilon}\big)$$

for any arbitrary algebraic number field of degree $d\ge 2$, where α is the residue of $\zeta(s,E)$ at its simple pole $s=1$. It is hard to improve Landau's result. Until 1993, for any algebraic number field of degree $d\ge 3$, Nowak[11] obtained the best result

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$$\sum_{n\le x} a(n)=\alpha x+\begin{cases}O\big(x^{1-\frac{2}{d}+\frac{8}{d(5d+2)}}(\log x)^{\frac{10}{5d+2}}\big) & \text{for } 3\le d\le 6,\\ O\big(x^{1-\frac{2}{d}+\frac{3}{2d^2}}(\log x)^{\frac{2}{d}}\big) & \text{for } d\ge 7.\end{cases}$$

In [1], Chandrasekharan and Good studied the l-th integral power sum of $a(n)$ in some Galois fields, and they showed that if E is a Galois extension of $\mathbb{Q}$ of degree d, then for any $\varepsilon>0$ and any integer $d\ge 2$, we have

$$\sum_{n\le x}(a(n))^l = xQ_l(\log x)+O\big(x^{1-\frac{2}{d^l}+\varepsilon}\big),$$

where $Q_l(t)$ is a suitable polynomial in t of degree $d^{l-1}-1$. Recently, Lü and Wang[9] improved the classical result of Chandrasekharan and Good by replacing $\frac{2}{d^l}$ in the error term with $\frac{3}{d^l+6}$.

In 2011, Lü and Yang[10] studied the asymptotic behavior of $(a(n^2))^l$ in some Galois fields. They proved that

Theorem A Suppose that E is a Galois extension of $\mathbb{Q}$ of degree $d\ge 3$. When d is odd, we have

$$\sum_{n\le x}(a(n^2))^l = xQ_m(\log x)+O\big(x^{1-\frac{3}{d^l+6}+\varepsilon}\big)$$

for any $\varepsilon>0$ and any integer $l\ge 1$, where $m=(\frac{d+1}{2})^l d^{l-1}$, $Q_m(t)$ is a suitable polynomial in t of degree $m-1$.

By the same method, Lü and Yang[10] also considered the k-dimensional divisor problem in some Galois fields over square numbers. Define

$$\tau_k^E(n)=\sum_{N(\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_k)=n}1. \tag{2}$$

In [10], it is also proved that

Theorem B Suppose that E is a Galois extension of $\mathbb{Q}$ of degree d. When d is odd, we have

$$\sum_{n\le x}\tau_k^E(n^2) = xQ_m(\log x)+O\big(x^{1-\frac{3}{md+6}+\varepsilon}\big)$$

for any $\varepsilon>0$ and any integer $k\ge 2$, where $m=\frac{k^2d+k}{2}$, $Q_m(t)$ is a suitable polynomial in t of degree $m-1$.

Theorem B constitutes an improvement upon the main terms of Deza and Varukhina's results (see [3]), and generalizes their results in some Galois fields.

Motivated by [4–6], the purpose of this note is to investigate the remainder terms in mean square, and we shall prove the following results.

Theorem 1.1 Subject to the conditions in Theorem A, define

$$\Delta(x):=\sum_{n\le x}(a(n^2))^l - xQ_m(\log x).$$

Then we have

$$\int_1^X \Delta^2(x)\mathrm{d}x \ll_\varepsilon X^{3-\frac{6}{md+3}+\varepsilon}$$

for any $\varepsilon > 0$.

Theorem 1.2 Subject to the conditions in Theorem B, define

$$R(x) := \sum_{n\le x} \tau_k^E(n^2) - xQ_m(\log x).$$

Then we have

$$\int_1^X R^2(x)\mathrm{d}x \ll_\varepsilon X^{3-\frac{6}{md+3}+\varepsilon}$$

for any $\varepsilon > 0$.

Theorem 1.3 Under the assumption of Theorem 1.1, for any Abelian polynomial $f(x)$ if

$$\Delta_1(x) := \sum_{m\le x} \rho(m) - C(f)x$$

where $C(f) = \mathrm{Res}_{s=1} \sum_{m=1}^{\infty} \frac{\rho(m)}{m^s}$, then we have

$$\int_1^X \Delta_1^2(x)\mathrm{d}x \ll_\varepsilon X^{3-\frac{3}{l}+\varepsilon}.$$

Notations As usual, $\omega(n)$ is the number of distinct prime divisors of n, and $\tau(n)$ is the divisor function. The Vinogradov symbol $A \ll B$ means that B is positive and the ratio $\frac{A}{B}$ is bounded. The letter ε denotes an arbitrary small positive number, not the same at each occurrence.

2 Some Lemmas

To prove our theorems, we need the following lemmas.

Lemma 2.1 Let $E/\mathbb{Q}$ be a Galois extension of degree d which is odd. $a(n)$ and $D_{2,l}(s)$ are defined in (1) and (6), respectively. Then we have

$$D_{2,l}(s) = \zeta^m(s,E) \cdot A_1(s) \tag{3}$$

for any integer $l \ge 1$, where $m = (\frac{d+1}{2})^l d^{l-1}$, $A_1(s)$ denotes a Dirichlet series, which is absolutely and uniformly convergent for $\sigma > \frac{1}{2}$.

Proof This follows immediately from (3.5) in [10, pp. 1929–1930]. □

Lemma 2.2 Let $E/\mathbb{Q}$ be a Galois extension of degree d which is odd. $\tau_k^E(n)$ and $D_{k,2}^E(s)$ are defined in (2) and (17), respectively. Then we have

$$L_{2,k}^E(s) = \zeta^m(s,E) \cdot A_2(s) \tag{4}$$

for any integer $k \ge 2$, where $m = \frac{k^2d+k}{2}$, $A_2(s)$ denotes a Dirichlet series, which is absolutely and uniformly convergent for $\sigma > \frac{1}{2}$.

Proof By replacing [9, (4.7)] with [9, (4.6)], this is similar to the proof of [9, (4.11)]. □

Lemma 2.3 Let E be an algebraic number field of degree d, then

$$\zeta(\sigma+\mathrm{i}t,E)\ll(1+|t|)^{\frac{d}{3}(1-\sigma)+\varepsilon}$$

for $\frac{1}{2}\le\sigma\le1+\varepsilon$ and any fixed $\varepsilon>0$.

Proof By [10, Lemma 2.5] and the Phragmén-Lindelöf principle for a strip (see [7, Theorem 5.53]), this lemma follows. □

3 Proof of Theorem 1.1

Let E be a Galois extension of $\mathbb{Q}$ of degree $d\ge3$ which is odd. Recall that $a(n)$ denotes the number of integral ideals in E with norm n, and

$$\zeta(s,E)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s},\quad s=\sigma+\mathrm{i}t,\quad \sigma>1.$$

Then by [2, Lemma 9], it is known that

$$a(n)\ll n^{\varepsilon}. \tag{5}$$

Therefore as in [10] we introduce the L-function associated to $a(n^2)$,

$$D_{2,l}(s)=\sum_{n=1}^{\infty}\frac{a(n^2)^l}{n^s},\quad \sigma>1. \tag{6}$$

From (5), we know that the Dirichlet series $D_{2,l}(s)$ is absolutely convergent in the half-plane $\sigma>1$.

Let $T=X^{\frac{3}{md+3}}$. From (5), (6) and Perron's formula (see [7, Proposition 5.54]), we get

$$\sum_{n\le x}a(n^2)^l=\frac{1}{2\pi\mathrm{i}}\int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T}D_{2,l}(s)\frac{x^s}{s}\mathrm{d}s+O\biggl(\frac{x^{1+\varepsilon}}{T}\biggr). \tag{7}$$

By the property that $D_{2,l}(s)$ only has a simple pole at $s=1$ for $\sigma>\frac{1}{2}$ and Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\le x}a(n^2)^l&=\frac{1}{2\pi\mathrm{i}}\left\{\int_{\frac12+\varepsilon-\mathrm{i}T}^{\frac12+\varepsilon+\mathrm{i}T}+\int_{\frac12+\varepsilon+\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T}+\int_{1+\varepsilon-\mathrm{i}T}^{\frac12+\varepsilon-\mathrm{i}T}\right\}D_{2,l}(s)\frac{x^s}{s}\mathrm{d}s+\operatorname*{Res}_{s=1}D_{2,l}(s)x+O\biggl(\frac{x^{1+\varepsilon}}{T}\biggr)\\&:=xQ_m(\log x)+J_1(x)+J_2(x)+J_3(x)+O(x^{1+\varepsilon}T^{-1})\\&:=xQ_m(\log x)+\Delta(x),\end{aligned} \tag{8}$$

where $Q_m(t)$ is a suitable polynomial in t of degree $m-1$. One can find the details of deducing (8) (see [10, p. 1929]).

In order to prove Theorem 1.1, it is enough to prove the following results:

$$\int_1^X J_i^2(x)\mathrm{d}x\ll_{\varepsilon}X^{3-\frac{6}{md+3}+\varepsilon},\quad i=1,2,3 \tag{9}$$

and
$$\int_1^X \left(O\left(\frac{x^{1+\varepsilon}}{T}\right)\right)^2 \mathrm{d}x \ll_\varepsilon X^{3-\frac{6}{md+3}+\varepsilon}. \tag{10}$$

It is easy to get
$$\int_1^X \left(O\left(\frac{x^{1+\varepsilon}}{T}\right)\right)^2 \mathrm{d}x = O\left(\frac{X^{3+\varepsilon}}{T^2}\right) \ll X^{3-\frac{6}{md+3}+\varepsilon}. \tag{11}$$

Now we consider the integral $J_1(x)$. We have
$$J_1(x) = \frac{1}{2\pi}\int_{-T}^{T} D_{2,l}\left(\frac{1}{2}+\varepsilon+\mathrm{i}t\right)\frac{x^{\frac{1}{2}+\varepsilon+\mathrm{i}t}}{\frac{1}{2}+\varepsilon+\mathrm{i}t}\mathrm{d}t.$$

Then
$$\begin{aligned}\int_1^X J_1^2(x)\mathrm{d}x &= \frac{1}{4\pi^2}\int_1^X \left(\int_{-T}^{T} D_{2,l}\left(\frac{1}{2}+\varepsilon+\mathrm{i}t_1\right)\frac{x^{\frac{1}{2}+\varepsilon+\mathrm{i}t_1}}{\frac{1}{2}+\varepsilon+\mathrm{i}t_1}\mathrm{d}t_1\right.\\ &\quad\left.\times\int_{-T}^{T}\overline{D_{2,l}\left(\frac{1}{2}+\varepsilon+\mathrm{i}t_2\right)}\frac{x^{\frac{1}{2}+\varepsilon-\mathrm{i}t_2}}{\frac{1}{2}+\varepsilon-\mathrm{i}t_2}\mathrm{d}t_2\right)\mathrm{d}x\\ &= \frac{1}{4\pi^2}\int_{-T}^{T}\int_{-T}^{T}\frac{D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_1)\overline{D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_2)}}{(\frac{1}{2}+\varepsilon+\mathrm{i}t_1)(\frac{1}{2}+\varepsilon-\mathrm{i}t_2)}\\ &\quad\times\int_1^X x^{1+2\varepsilon+\mathrm{i}(t_1-t_2)}\mathrm{d}x\mathrm{d}t_1\mathrm{d}t_2\\ &\ll X^{2+2\varepsilon}\int_{-T}^{T}\int_{-T}^{T}\frac{|D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_1)||D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_2)|}{(1+|t_1|)(1+|t_2|)(1+|t_1-t_2|)}\mathrm{d}t_2\mathrm{d}t_1\\ &\ll X^{2+2\varepsilon}\int_{-T}^{T}\int_{-T}^{T}\left(\frac{|D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_1)|^2}{(1+|t_1|)^2}+\frac{|D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_2)|^2}{(1+|t_2|)^2}\right)\\ &\quad\times\frac{1}{1+|t_1-t_2|}\mathrm{d}t_2\mathrm{d}t_1\\ &\ll X^{2+2\varepsilon}\int_{-T}^{T}\frac{|D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_1)|^2}{(1+|t_1|)^2}\mathrm{d}t_1\int_{-T}^{T}\frac{\mathrm{d}t_2}{1+|t_1-t_2|}.\end{aligned} \tag{12}$$

To go further, we get
$$\begin{aligned}\int_{-T}^{T}\frac{\mathrm{d}t_2}{1+|t_1-t_2|} &\ll \int_{t_1-1}^{t_1+1}\mathrm{d}t_2 + \left(\int_{t_1+1}^{T}+\int_{-T}^{t_1-1}\right)\frac{\mathrm{d}t_2}{|t_1-t_2|}\\ &\ll 1+\int_{t_1+1}^{T}\frac{\mathrm{d}t_2}{|t_1-t_2|}\\ &\ll \int_1^{T+|t_1|}\frac{\mathrm{d}t}{t} \ll \log 2T.\end{aligned} \tag{13}$$

By (12)–(13),
$$\int_1^X J_1^2(x)\mathrm{d}x \ll X^{2+2\varepsilon}\log 2T\int_{-T}^{T}\frac{|D_{2,l}(\frac{1}{2}+\varepsilon+\mathrm{i}t_1)|^2}{(1+|t_1|)^2}\mathrm{d}t_1. \tag{14}$$

From this, Lemmas 2.1 and 2.3, we have (for $d \geq 3$)

$$\begin{aligned}\int_1^X J_1^2(x)\mathrm{d}x &\ll X^{2+3\varepsilon} + X^{2+3\varepsilon}\int_1^T \left|\zeta^m\left(\frac{1}{2}+\varepsilon+\mathrm{i}t, E\right)A_1\left(\frac{1}{2}+\varepsilon+\mathrm{i}t_1\right)\right|^2 t^{-2}\mathrm{d}t \\ &\ll X^{2+3\varepsilon} + X^{2+3\varepsilon}\int_1^T \left|\zeta^m\left(\frac{1}{2}+\varepsilon+\mathrm{i}t, E\right)\right|^2 t^{-2}\mathrm{d}t \\ &\ll X^{2+3\varepsilon} + X^{2+3\varepsilon}\int_1^T \left(t^{\frac{md}{6}+\varepsilon}\right)^2 t^{-2}\mathrm{d}t \\ &\ll X^{2+3\varepsilon} + X^{2+4\varepsilon}T^{\frac{md}{3}-1} \\ &\ll X^{3-\frac{6}{md+3}+\varepsilon}. \end{aligned} \tag{15}$$

Finally we estimate trivial bounds of the integrals $J_2(x)$ and $J_3(x)$. By Lemma 2.3, we get

$$\begin{aligned} J_2(x)+J_3(x) &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^\sigma|\zeta^m(\sigma+\mathrm{i}T, E)|T^{-1}\mathrm{d}\sigma \\ &\ll \max_{\frac{1}{2}+\varepsilon\leq\sigma\leq 1+\varepsilon} x^\sigma T^{\frac{md}{3}(1-\sigma)+\varepsilon}T^{-1} \\ &= \max_{\frac{1}{2}+\varepsilon\leq\sigma\leq 1+\varepsilon}\left(\frac{x}{T^{\frac{md}{3}}}\right)^\sigma T^{\frac{md}{3}-1+\varepsilon} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{md}{6}-1+\varepsilon}, \end{aligned}$$

which yields

$$\begin{aligned}\int_1^X (J_2(x)+J_3(x))^2\mathrm{d}x &\ll \int_1^X \left(\frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{md}{6}-1+\varepsilon}\right)^2\mathrm{d}x \\ &\ll \int_1^X \left(\frac{x^{1+\varepsilon}}{T}\right)^2\mathrm{d}x + \int_1^X \left(x^{\frac{1}{2}+\varepsilon}T^{\frac{md}{6}-1+\varepsilon}\right)^2\mathrm{d}x \\ &\ll \frac{X^{3+\varepsilon}}{T^2} + X^{2+2\varepsilon}T^{\frac{md}{3}-2+2\varepsilon} \\ &\ll X^{3-\frac{6}{md+3}+\varepsilon}. \end{aligned} \tag{16}$$

The inequalities (9) and (10) immediately follow from (3), (10), (15) and (16). That is,

$$\int_1^X \Delta^2(x)\mathrm{d}x \ll_\varepsilon X^{3-\frac{6}{md+3}+\varepsilon}.$$

Then this completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

By the definition of $\tau_k^E(n)$ in (2) we can define an L-function associated to the function $\tau_k^E(n^2)$,

$$L_{k,2}^E(s) = \sum_{n=1}^\infty \frac{\tau_k^E(n^2)}{n^s} \quad (\sigma > 1), \tag{17}$$

which is absolutely convergent in this region.

From Lemma 2.2, we know that $L_{k,2}^{E}(s)$ admits a meromorphic continuation to the half-plane $\sigma > \frac{1}{2}$ and only has a pole $s = 1$ of order $m = \frac{k^2d+k}{2}$ in this region. By Lemma 2.2, the detail of the proof of Theorem 1.2 is similar to that of Theorem 1.1. Hence, we omit it here.

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References

[1] Chandrasekharan, K. and Good, A., On the number of integral ideals in Galois extensions, *Monatsh. Math.*, 1983, 95(2): 99-109.

[2] Chandrasekharan, K. and Narasimhan, R., The approximate functional equation for a class of zeta-functions, *Math. Ann.* 1963, 152(1): 30-64.

[3] Deza, E. and Varukhina, L., On mean values of some arithmetic functions in number fields, *Discrete Math.*, 2008, 308(21): 4892-4899.

[4] Heath-Brown, D.R., The number of Abelian groups of order at most x, *Astérisque*, 1991, 198/199/200: 153-163.

[5] Ivić, A., The number of finite non-isomorphic abelian groups in mean square, *Hardy-Ramanujan J.*, 1986, 9: 17-23.

[6] Ivić, A., On the error term for the counting functions of finite abelian groups, *Monatsh. Math.*, 1992, 114(2): 115-124.

[7] Iwaniec, H. and Kowalski, E., Analytic Number Theory, Amer. Math. Soc. Coll. Publ., Vol. 53, 2004, 204-216.

[8] Landau, E., Einführung in die elementare and analytische theorie der algebraischen zahlen und der ideale, Leipzig: Teubner, 1927 (in German).

[9] Lü, G.S. and Wang, Y.H., Note on the number of integral ideals in Galois extension, *Sci. China Math.*, 2010, 53(9): 2417-2424.

[10] Lü, G.S. and Yang, Z.S., The average behavior of the coefficients of Dedekind zeta function over square numbers, *J. Number Theory*, 2011, 131(10): 1924-1938.

[11] Nowak, W.G., On the distribution of integral ideals in algebraic number fields, *Math. Nachr.*, 1993, 161(1): 59-74.

关于平方数的戴德金 ζ 函数系数的和式余项

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摘要：设 E 是有理数域 $\mathbb{Q}$ 上的一个代数数域. $a(n)$ 为 E 上范数 n 的整理想的个数. 再设 $\Delta(x)$ 为和式 $\sum_{n\leq x}(a(n^2))^l$ 渐近式的余项. 本文利用解析方法得到了

$$\int_1^X \Delta^2(x)\mathrm{d}x$$

的一个比较好的上界. 该结果在均值上改进了吕广世等人 [*J. Number Theory*, 2011, 131: 1924-1938] 所得的结果.

关键词：戴德金 ζ 函数; 代数数域; 平方数