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Some Results on the Sum of the Normalized Laplacian Eigenvalues of Graphs

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Abstract: For a connected graph G, let $L(G)$ and $\mathcal{L}(G)$ be its Laplacian matrix and normalized Laplacian matrix, respectively. Suppose that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ are the Laplacian eigenvalues and normalized Laplacian eigenvalues of G, respectively. In this paper, we give three new lower bounds on λ_1. The first two bounds are both stronger than Das et al.'s lower bound in [*Ars Combin.*, 2015, 118: 143-154], and the last is stronger than Zhang's lower bound in [*Ars Combin.*, 2004, 72: 191-198]. In addition, we provide inequalities on the normalized Laplacian eigenvalues using the degrees of G. At the same time, we focus on the relationship between the Laplacian eigenvalues and the normalized Laplacian eigenvalues of G.

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0 Introduction

Spectral graph theory has a long history. In the early days, matrix theory and linear algebra were used to analyze adjacency matrices of graphs. Algebraic methods have been proven to be especially effective in graph research. The study of graph eigenvalues shows rich connections among many other areas of mathematics.

Let $G = (V, E)$ be a simple (connected) graph with the vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by $d(v_i)$ (or d_i), is the number of vertices adjacent to v_i. Without loss of generality, we may suppose $d_1 \geq d_2 \geq \cdots \geq d_n$ throughout the paper. We often use the notation $v_i \sim v_j$ (or $i \sim j$) to mean that v_i (or i) is adjacent to v_j (or j) in G. The adjacency matrix $A(G)$ (or A) of a graph $G = (V, E)$ is defined to have entries

$$A_{ij} = \begin{cases} 0, & \text{if } i \nsim j; \\ 1, & \text{if } i \sim j. \end{cases}$$

Let $D = D(G) = \operatorname{diag}\{d_1, d_2, \cdots, d_n\}$ be a diagonal matrix of degrees in G. Then the Laplacian matrix $L(G)$ (or L) of a graph $G = (V, E)$ is defined by $L = D - A$. The normalized Laplacian

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$\mathcal{L}(G)$ (or $\mathcal{L}$) of a graph $G = (V, E)$ is defined by $\mathcal{L}(G) = D^{-\frac{1}{2}} L(G) D^{-\frac{1}{2}}$, that is, $\mathcal{L}$ has entries

$$\mathcal{L}_{ij} = \begin{cases} 1, & \text{if } i = j; \\ -\dfrac{1}{\sqrt{d_i d_j}}, & \text{if } i \sim j; \\ 0, & \text{otherwise.} \end{cases}$$

It is known that both of L and $\mathcal{L}$ are positive semidefinite matrices. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ be the Laplacian eigenvalues and the normalized Laplacian eigenvalues of G, respectively.

The adjacency matrix and the Laplacian matrix have been more widely investigated than the normalized Laplacian matrix. One reason is that the normalized Laplacian is a rather new tool which has been popularized by Chung[1]. In some situations, the normalized Laplacian matrix is a more natural tool that works better than the adjacency matrix and Laplacian matrix. We can obtain much information from the normalized Laplacian eigenvalues of graphs.

Generally speaking, half of the main problems of spectral theory lie in deriving bounds of eigenvalues. There are many rich results on the bound of the largest Laplacian eigenvalue of graphs[3, 7−8]. However, there are not many results on the largest normalized Laplacian eigenvalue of graphs[1−2, 4, 9]. In [1], Chung proved

Lemma 0.1[1] Let G be a connected graph of order $n \geq 2$. Then

(1) $\lambda_1 \geq \frac{n}{n-1}$ and equality holds if and only if $G \cong K_n$;

(2) $\lambda_{n-1} \leq \frac{n}{n-1}$ with equality holding if and only if $G \cong K_n$. If G is not the complete graph K_n, then $\lambda_{n-1} \leq 1$;

(3) for all $1 \leq i \leq n-1$, we have $\lambda_i \leq 2$ with $\lambda_1 = 2$ if and only if G is bipartite.

From Lemma 0.1, we know that the upper bound on λ_1 is equal to 2. So we are interested in the lower bound on λ_1. In this paper, we present three new lower bounds on λ_1. The first two bounds are both stronger than Das' lower bound in [4], and the last new bound is stronger than Zhang's bound in [9].

In addition, Grone[6] proved that if G is a connected graph with n vertices, then $\sum_{i=1}^{k} \mu_i \geq 1 + \sum_{i=1}^{k} d_i$ for $1 \leq k \leq n-1$. We are stimulated by Grone's result and provide inequalities between the normalized Laplacian eigenvalues and the degrees of G. Meanwhile, we consider the relationship between $\sum_{i=1}^{k} \lambda_i$ and $\sum_{i=1}^{k} \mu_i$ of a connected graph G for $1 \leq k \leq n-1$.

1 Largest Normalized Laplacian Eigenvalue of Graphs

First, we discuss some lower bounds on λ_1 (the largest normalized Laplacian eigenvalue) of connected graphs. Recently, Das et al. proved the following lower bound on λ_1.

Lemma 1.1[4] Let G be a connected graph with n vertices. Then

$$\lambda_1 \geq 1 + \sqrt{\frac{2}{n(n-1)} \sum_{i \sim j} \frac{1}{d_i d_j}}.$$

Moreover, equality holds if and only if $G \cong K_n$.

We now introduce three new lower bounds on λ_1.

Theorem 1.1 Let G be a connected graph with n vertices. Then

$$\lambda_1 \geq \sqrt{\frac{n}{n-1} + \frac{2}{n-1}\sum_{i\sim j}\frac{1}{d_i d_j}}.$$

Moreover, equality holds if and only if $G \cong K_n$.

Proof Considering the trace of the matrix $\mathcal{L}^2$, we have

$$(n-1)\lambda_1^2 \geq \sum_{i=1}^{n}\lambda_i^2 = \mathrm{tr}(\mathcal{L}^2) = n + 2\sum_{i\sim j}\frac{1}{d_i d_j},$$

where equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \frac{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}{n-1} = \frac{n}{n-1}$. Then the second part of the theorem follows from Lemma 1.1. □

Remark 1.1 Theorem 1.1 is stronger than Lemma 1.1. In fact, $\sqrt{\frac{n}{n-1} + \frac{2}{n-1}\sum_{i\sim j}\frac{1}{d_i d_j}} \geq 1 + \sqrt{\frac{2}{n(n-1)}\sum_{i\sim j}\frac{1}{d_i d_j}}$ is equivalent to $(\sum_{i\sim j}\frac{1}{d_i d_j} - \frac{n}{2(n-1)})^2 \geq 0$.

Theorem 1.2 Let G be a connected graph with n vertices. Then

$$\lambda_1 \geq 1 + \frac{2}{n}\sum_{i\sim j}\frac{1}{d_i d_j}.$$

Moreover, equality holds if and only if $G \cong K_n$.

Proof We consider the trace of the matrix $(\mathcal{L} - xI_n)^2$ (I_n is an identity diagonal matrix of order n) with $x = \frac{\lambda_1}{2}$,

$$\mathrm{tr}(\mathcal{L} - xI_n)^2 = n\left(1 - \frac{\lambda_1}{2}\right)^2 + 2\sum_{i\sim j}\frac{1}{d_i d_j}. \tag{1.1}$$

On the other hand, since $(\mathcal{L} - xI_n)^2$ has eigenvalues $(\lambda_1 - x)^2, (\lambda_2 - x)^2, \cdots, (\lambda_n - x)^2$, we have $\mathrm{tr}(\mathcal{L} - xI_n)^2 = \sum_{i=1}^{n}(\lambda_i - \frac{\lambda_1}{2})^2$. Since

$$-\frac{\lambda_1}{2} \leq 0 - \frac{\lambda_1}{2} \leq \lambda_i - \frac{\lambda_1}{2} \leq \lambda_1 - \frac{\lambda_1}{2} = \frac{\lambda_1}{2},$$

we have

$$\mathrm{tr}(\mathcal{L} - xI_n)^2 = \sum_{i=1}^{n}\left(\lambda_i - \frac{\lambda_1}{2}\right)^2 \leq n\left(\frac{\lambda_1}{2}\right)^2. \tag{1.2}$$

Combining (1.1) and (1.2), we obtain $\lambda_1 \geq 1 + \frac{2}{n}\sum_{i\sim j}\frac{1}{d_i d_j}$, where equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \frac{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}{n-1} = \frac{n}{n-1}$ for a connected graph G. □

Remark 1.2 Note that

$$\sum_{i\sim j}\frac{1}{d_i d_j} = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{d_i}\sum_{j:j\sim i}\frac{1}{d_j} \geq \frac{1}{2}\sum_{i=1}^{n}\frac{1}{d_i}\sum_{j:j\sim i}\frac{1}{n-1} = \frac{n}{2(n-1)}.$$

Then one can check that Theorem 1.2 is stronger than Theorem 1.1. In fact, we know that $1 + \frac{2}{n}\sum_{i\sim j}\frac{1}{d_i d_j} \geq \sqrt{\frac{n}{n-1} + \frac{2}{n-1}\sum_{i\sim j}\frac{1}{d_i d_j}}$ is equivalent to $\frac{2(n-2)}{n(n-1)}\sum_{i\sim j}\frac{1}{d_i d_j} + \frac{4}{n^2}(\sum_{i\sim j}\frac{1}{d_i d_j})^2 - \frac{1}{n-1} \geq 0$ which is true for $\sum_{i\sim j}\frac{1}{d_i d_j} \geq \frac{n}{2(n-1)}$.

We can view the eigenvectors g of $\mathcal{L}(G)$ as functions which assign to each vertex v_i of G a real value $g(i)$. In particular, if $V=\{v_1,v_2,\cdots,v_n\}$ and $g=(g(1),g(2),\cdots,g(n))^{\mathrm{T}}$, then g can be viewed as a function which assigns to each vertex v_i the real value $g(i)$. By letting $g=D^{\frac{1}{2}}f$, we have

$$\frac{g^{\mathrm{T}}\mathcal{L}g}{g^{\mathrm{T}}g}=\frac{f^{\mathrm{T}}D^{\frac{1}{2}}\mathcal{L}D^{\frac{1}{2}}f}{(D^{\frac{1}{2}}f)^{\mathrm{T}}D^{\frac{1}{2}}f}=\frac{f^{\mathrm{T}}Lf}{f^{\mathrm{T}}Df}=\frac{\sum_{i\sim j}(f(i)-f(j))^2}{\sum_i f^2(i)d_i}.$$

In 2004, Zhang[9] proved that $\lambda_1\geq 1+\frac{1}{d_1}$. Using an idea from [1, Section 2.3], we can improve Zhang's bound on λ_1.

Theorem 1.3 Let G be a connected graph with n vertices. Then

$$\lambda_1\geq 1+\frac{1}{d_2}.$$

Proof Let v_1 be the vertex with $d(v_1)=d_1$. We can define f as follows:

$$f(u)=\begin{cases}\sum_{i:i\sim 1}d_i, & \text{if } u=v_1;\\ -d_1, & \text{if } u\sim v_1;\\ 0, & \text{otherwise.}\end{cases}$$

Note that with the choice of f, we have $(D\mathbf{1})^{\mathrm{T}}f=0$ ($\mathbf{1}$ denotes the all ones vector) and

$$\begin{aligned}\lambda_1&\geq\frac{\sum_{i\sim j}(f(i)-f(j))^2}{\sum_i f^2(i)d_i}\\&\geq\frac{d_1(\sum_{i:i\sim 1}d_i+d_1)^2}{d_1(\sum_{i:i\sim 1}d_i)^2+d_1^2\sum_{i:i\sim 1}d_i}\\&=\frac{\sum_{i:i\sim 1}d_i+d_1}{\sum_{i:i\sim 1}d_i}\\&=1+\frac{d_1}{\sum_{i:i\sim 1}d_i}.\end{aligned}$$

Since v_1 has the largest degree d_1, other vertices have degrees at most d_2. Thus $\sum_{i:i\sim 1}d_i\leq\sum_{i:i\sim 1}d_2=d_1d_2$, from which Theorem 1.3 follows. □

Remark 1.3 While Theorem 1.2 is stronger than Theorem 1.1 (see Remark 1.2). Generally speaking, Theorem 1.3 is not comparable with either Theorem 1.1 or Theorem 1.2. When G is the complete graph K_n, all three bounds are equal to $\frac{n}{n-1}$. Let G_1 be the graph with n vertices obtained from cycle C_{n-1} by adding a pendent edge and G_2 be the graph with n vertices obtained by cycle C_n adding an edge which does not exist in C_n.

If $n\geq 3$, then we have $\sqrt{\frac{9n-1}{6(n-1)}}<\frac{9n-1}{6n}<\frac{3}{2}$. Table 1 below with the graph G_1 shows that Theorem 1.3 is the strongest.

If $3\leq n\leq 4$, then we have $\frac{4}{3}<\sqrt{\frac{27n-8}{18(n-1)}}<\frac{27n-8}{18n}$. Table 1 below with the graph G_2 shows that Theorem 1.3 is the weakest.

If $n\geq 5$, then we have $\sqrt{\frac{27n-8}{18(n-1)}}<\frac{4}{3}<\frac{27n-8}{18n}$. Table 1 below with the graph G_2 shows that Theorem 1.1 is the weakest.

Table 1 A comparison among Theorems 1.1–1.3

Graph	Theorem 1.1	Theorem 1.2	Theorem 1.3
G_1	$\sqrt{\frac{9n-1}{6(n-1)}}$	$\frac{9n-1}{6n}$	$\frac{3}{2}$
G_2	$\sqrt{\frac{27n-8}{18(n-1)}}$	$\frac{27n-8}{18n}$	$\frac{4}{3}$

2 Normalized Laplacian Eigenvalues and Degrees of Graphs

This section is devoted to investigating the lower bound on $\sum_{i=1}^{k}\lambda_i$ $(1\le k\le n-1)$ for a connected graph G. The following lemmas are needed to prove our results.

Lemma 2.1 Let B be a $k\times k$ $(k<n)$ principal submatrix of $\mathcal{L}(G)$ where G is a connected graph with n vertices. Then all the diagonal entries of B^{-1} are at least 1.

Proof Let B be a $k\times k$ $(k<n)$ principal submatrix of $\mathcal{L}(G)$. Since $\mathcal{L}(G)$ and $L(G)$ are positive diagonal, then B' is congruent with a $k\times k$ $(k<n)$ principal submatrix of $L(G)$. If $k=n-1$, G is connected, then $\det(B')=$ the number of spanning trees of G, that is to say $\det(B')>0$. So B' is positive definite, and B is also positive definite. Then all $k\times k$ $(k\le n-1)$ principal submatrix of $\mathcal{L}(G)$ are positive definite.

Since $\mathcal{L}(G)$ is an M-matrix, then B is a reversible M-matrix. And $B=I-T$, which is positive definite, then $0\le\lambda_1(T)<1$,

$$B^{-1}=(I-T)^{-1}=I+T+T^2+\cdots,\quad \text{if } n\to\infty,\ T^n\to 0,$$

so all the diagonal entries of B^{-1} are at least 1. □

Theorem 2.1 Let G be a connected graph with n vertices, and $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_{n-1}\ge\lambda_n=0$ be the normalized Laplacian eigenvalues of $\mathcal{L}(G)$. Then for any $1\le k\le n-1$ and for any partition $(X,\overline{X})$ of $V(G)$,

$$\sum_{i=1}^{k}\lambda_i\ge k+\sum_{ij\in E(X,\overline{X})}\frac{1}{d_id_j}.$$

Proof Let $X\subseteq V$ and partition $\mathcal{L}(G)$ according to the vertex partition $(X,\overline{X})$ as follows:

$$\mathcal{L}(G)=\begin{pmatrix} B & C\\ C^{\mathrm{T}} & E\end{pmatrix},$$

where B is $k\times k$ principal submatrix of $\mathcal{L}(G)$. Since G is connected, we have $C\ne 0$. By Lemma 2.1, we know that both of B and B^{-1} are positive definite matrices and all the entries of B^{-1} are nonnegative. Notice that $\mathcal{L}=FF^{\mathrm{T}}$, where

$$F=\begin{pmatrix} B^{\frac12} & 0\\ C^TB^{-\frac12} & G\end{pmatrix}$$

and $G=(E-C^{\mathrm{T}}B^{-1}C)^{\frac{1}{2}}$. Then $\mathcal{L}$ has the same eigenvalues as $K=F^{\mathrm{T}}F$, where

$$K=\begin{pmatrix} B+B^{-\frac{1}{2}}CC^{\mathrm{T}}B^{-\frac{1}{2}} & * \\ * & * \end{pmatrix}.$$

By Lemma 2.1, all entries of B^{-1} are nonnegative and all the diagonal entries of B^{-1} are at least 1. These imply that

$$\mathrm{tr}(B^{-\frac{1}{2}}CC^{\mathrm{T}}B^{-\frac{1}{2}})=\mathrm{tr}(B^{-1}CC^{\mathrm{T}})\ge\sum_{i=1}^{k}\sum_{\substack{j:j\sim i\\ k+1\le j\le n}}\frac{1}{d_id_j}.$$

And thus

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i&\ge\mathrm{tr}(B)+\mathrm{tr}(B^{-1}CC^{\mathrm{T}})\\ &\ge k+\sum_{i=1}^{k}\sum_{\substack{j:j\sim i\\ k+1\le j\le n}}\frac{1}{d_id_j}\\ &=k+\sum_{ij\in E(X,\overline{X})}\frac{1}{d_id_j}.\end{aligned}$$ □

Using Lemma 2.1 and substituting $k=1$ in Theorem 2.1, we have the following corollary which was first proved by Chung[1].

Corollary 2.1[1] $\lambda_1\ge\frac{n}{n-1}$.

From Theorem 2.1, we can also have the following result.

Corollary 2.2 Let $\overline{d}$ be the average degree of a graph G. Then

$$\lambda_1\ge 1+\frac{1}{\overline{d}}.$$

Proof Let $X=\{v_i\}$ contain any single vertex v_i. Then $E(X,\overline{X})$ contains all edges incident to v_i. By Theorem 2.1, we have

$$\lambda_1\ge 1+\frac{1}{d_i}\sum_{j:j\sim i}\frac{1}{d_j}.$$

We evaluate the sum $\sum_i\sum_{j:j\sim i}\frac{1}{d_j}$. On one hand,

$$\sum_i\sum_{j:j\sim i}\frac{1}{d_j}=\sum_j\frac{1}{d_j}\sum_{i:i\sim j}1=\sum_j\frac{1}{d_j}d_j=n.$$

On the other hand,

$$\sum_i\frac{d_i}{\overline{d}}=\frac{1}{\overline{d}}\sum_i d_i=\frac{1}{\overline{d}}n\overline{d}=n.$$

So

$$\sum_i\sum_{j:j\sim i}\frac{1}{d_j}=\sum_i\frac{d_i}{\overline{d}}.$$

By an averaging argument, there exists some i such that $\sum_{j:j\sim i}\frac{1}{d_j}\geq\frac{d_i}{\bar{\bar{d}}}$. Therefore

$$\lambda_1\geq 1+\frac{1}{d_i}\sum_{j:j\sim i}\frac{1}{d_j}\geq 1+\frac{1}{\bar{\bar{d}}}.$$

3 Sum Relation Between Laplacian Eigenvalues and Normalized Laplacian Eigenvalues

In this section, we discuss the relation between Laplacian eigenvalues and normalized Laplacian eigenvalues. First, we need some lemmas to prove the main theorem.

Lemma 3.1[5] If A and B are Hermitian matrices, then

$$\sum_{i=1}^{k}\lambda_i(A+B)\leq\sum_{i=1}^{k}\lambda_i(A)+\sum_{i=1}^{k}\lambda_i(B),$$

where $\lambda_i(\,\cdot\,)$ is the i-th largest eigenvalue of the indicated matrix.

Next, we consider the matrix $\mathcal{L}(G)-\frac{n}{2m}L(G):=\mathcal{L}-L'$, where m is the number of edges and n is the number of vertices of G, respectively. Let $\nu_1\geq\nu_2\geq\cdots\geq\nu_n$ be the eigenvalues of $\mathcal{L}-L'$.

Lemma 3.2 For each k with $1\leq k\leq n-1$,

$$\sum_{i=1}^{k}\nu_i\leq\sqrt{k(n-k)\left(\frac{n}{4m^2}\sum_{i=1}^{n}d_i^2+\frac{n}{m}+\frac{2m}{n\Delta^2}-\frac{\Delta+2}{\Delta}\right)},$$

where Δ is the maximum degree of G.

Proof Let us define $M_k=\sum_{i=1}^{k}\nu_i$ for convenience. Considering $(\mathcal{L}-L')^2$, since $\frac{1}{\sqrt{d_id_j}}\geq\frac{1}{\Delta}$ we have

$$\begin{aligned}\sum_{i=1}^{n}\nu_i^2&=\sum_{i=1}^{n}\left(1-\frac{n}{2m}d_i\right)^2+2\sum_{i\sim j}\left(\frac{n}{2m}-\frac{1}{\sqrt{d_id_j}}\right)^2\\&\leq\frac{n^2}{4m^2}\sum_{i=1}^{n}d_i^2+\frac{n^2}{2m}-n+2m\left(\frac{n}{2m}-\frac{1}{\Delta}\right)^2\\&=\frac{n^2}{4m^2}\sum_{i=1}^{n}d_i^2+\frac{n^2}{m}+\frac{2m}{\Delta^2}-\frac{n(\Delta+2)}{\Delta}.\end{aligned}$$

Since $\sum_{i=1}^{n}\nu_i=\mathrm{tr}(\mathcal{L}-L')=0$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}M_k^2&=(\nu_{k+1}+\nu_{k+2}+\cdots+\nu_n)^2\\&\leq(n-k)\big(\nu_{k+1}^2+\nu_{k+2}^2+\cdots+\nu_n^2\big)\\&=(n-k)\left(\sum_{i=1}^{n}\nu_i^2-\sum_{i=1}^{k}\nu_i^2\right)\\&\leq(n-k)\left(\sum_{i=1}^{n}\nu_i^2-\frac{1}{k}M_k^2\right).\end{aligned}$$

Solving for M_k,

$$M_k \le \sqrt{\frac{k(n-k)}{n}\sum_{i=1}^{n}\nu_i^2} \le \sqrt{\frac{k(n-k)}{n}\left(\frac{n^2}{4m^2}\sum_{i=1}^{n}d_i^2+\frac{n^2}{m}+\frac{2m}{\Delta^2}-\frac{n(\Delta+2)}{\Delta}\right)} = \sqrt{k(n-k)\left(\frac{n}{4m^2}\sum_{i=1}^{n}d_i^2+\frac{n}{m}+\frac{2m}{n\Delta^2}-\frac{\Delta+2}{\Delta}\right)}.$$

Finally, we give the main theorem in this section.

Theorem 3.1 Let G be a connected graph with n vertices and m edges and $L(G)$, $\mathcal{L}(G)$ be its Laplacian and normalized Laplacian matrices, respectively. Suppose that $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the Laplacian eigenvalues and normalized Laplacian eigenvalues of G, respectively. Then

$$\sum_{i=1}^{k}\lambda_i \le \frac{n}{2m}\sum_{i=1}^{k}\mu_i + \sqrt{k(n-k)\left(\frac{n}{4m^2}\sum_{i=1}^{n}d_i^2+\frac{n}{m}+\frac{2m}{n\Delta^2}-\frac{\Delta+2}{\Delta}\right)}.$$

Proof Since

$$\mathcal{L}(G) = \frac{n}{2m}L(G) + \left(\mathcal{L}(G) - \frac{n}{2m}L(G)\right) = L' + (\mathcal{L} - L'),$$

by Lemmas 3.1–3.2, we have

$$\sum_{i=1}^{k}\lambda_i \le \frac{n}{2m}\sum_{i=1}^{k}\mu_i + \sum_{i=1}^{k}\lambda_i(\mathcal{L}-L') \le \frac{n}{2m}\sum_{i=1}^{k}\mu_i + \sqrt{k(n-k)\left(\frac{n}{4m^2}\sum_{i=1}^{n}d_i^2+\frac{n}{m}+\frac{2m}{n\Delta^2}-\frac{\Delta+2}{\Delta}\right)}. \qquad \square$$

Remark 3.1 Since the Laplacian eigenvalues and normalized Laplacian eigenvalues of K_n are $\{n^{(n-1)}, 0\}$ and $\{(\frac{n}{n-1})^{(n-1)}, 0\}$ (with exponents denoting multiplicities), respectively. Equality in Theorem 3.1 holds for K_n.

For two non-increasing real sequences x and y of length n, we say that x is majorized by y (denoted by $x \preceq y$) if

$$\sum_{i=1}^{k}x_i \le \sum_{i=1}^{k}y_i \quad \text{for all } k \le n-1, \quad \text{and} \quad \sum_{i=1}^{n}x_i = \sum_{i=1}^{n}y_i.$$

We finish the paper with the following two conjectures.

Corollary 3.1 Let G be a connected graph. Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are normalized Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. Then

$$\{\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)\} \succeq \left\{1+\frac{1}{\overline{d}}, 1, 1, \cdots, 1, 1-\frac{1}{\overline{d}}\right\}.$$

Corollary 3.2 Let G be a connected graph. Suppose that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are Laplacian eigenvalues and normalized Laplacian eigenvalues, respectively. Then

$$\{\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)\} \preceq \left\{\frac{n}{2m}\mu_1(G), \frac{n}{2m}\mu_2(G), \cdots, \frac{n}{2m}\mu_n(G)\right\}.$$

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关于图的规范拉普拉斯特征值和的若干结果

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摘要： 对任意一个连通图 G, 记 $L(G)$ 和 $\mathcal{L}(G)$ 分别为 G 的拉普拉斯矩阵和规范拉普拉斯矩阵. 令 $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ 和 $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ 分别为 G 的拉普拉斯特征值和规范拉普拉斯特征值. 本文给出了 λ_1 的三个新的下界. 前两个下界优于 Das 等在 [*Ars Combin.*, 2015, 118: 143-154] 中给出的下界, 第三个下界优于张晓东在 [*Ars Combin.*, 2004, 72: 191-198] 中给出的下界. 另一方面讨论了规范拉普拉斯特征值与 G 的度序列之间的关系. 同时也讨论了图的拉普拉斯特征值和规范拉普拉斯特征值之间的关系.

关键词： 规范拉普拉斯; 拉普拉斯; 和; 下界