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Product Formula and Independence for Complex Multiple Wiener-Itô Integrals

CHEN Yong

(*School of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan, 411201, P. R. China*)

Abstract: We present a product formula for complex Wiener-Itô integrals. As an application, we show the Üstünel-Zakai independence criterion for complex multiple Wiener-Itô integrals.

Keywords: complex Wiener-Itô integrals; product formula; Üstünel-Zakai independence criterion

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0 Introduction

The product formula of real multiple Wiener-Itô integrals is well known. Using this formula there are many interesting findings such as Üstünel-Zakai independence criterion[8, 15] for two multiple Wiener-Itô integrals, Nourdin-Rosiński asymptotic moment-independence criterion between blocks consisting of multiple Wiener-Itô integrals[11] and Fourth Moment Theorem (or say: Nualart-Peccati criterion) of a normalized sequence of real multiple Wiener-Itô integrals in a fixed Wiener chaos[10, 13].

Both real multiple Wiener-Itô integrals and complex multiple Wiener-Itô integrals (see [7] or Section 1 below) were established by Itô almost at the same time[6−7]. However, the product formula of complex multiple Wiener-Itô integrals is still unknown up to now as far as we know. The key aim of this paper is to answer this question (see Theorem 2.1). As far as we know, there exist at least three different approaches to prove the product formula. In this paper, we adopt the most simple one by using the relationship between complex multiple Wiener-Itô integrals and complex Hermite polynomials given by Itô[7].

As applications, we will show Üstünel-Zakai independence criterion, i.e., a necessary and sufficient condition on the pair of kernels (f, g) is derived under which the complex multiple Wiener-Itô integrals $I_{a,b}(f)$, $I_{c,d}(g)$ are independent (see Theorem 3.1).

1 Preliminaries

In this section, we shortly recall the theory of complex multiple Wiener-Itô integrals of Itô[7]. Consider a triple $(T, \mathcal{B}, \mu)$, where the measure μ is positive, σ-finite and non-atomic.

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E-mail: zhishi@pku.org.cn

$\mathfrak{H} = L^2(T, \mathcal{B}, \mu)$ is a complex separable Hilbert space. A complex Gaussian random measure over $(T, \mathcal{B})$, with control μ, is a centered complex Gaussian family of the type

$$\mathbf{M} = \{M(B) : B \in \mathcal{B}, \mu(B) < \infty\},$$

such that, for every $B, C \in \mathcal{B}$ with finite measure,

$$E[M(B)\overline{M(C)}] = \mu(B \cap C).$$

Notation 1.1 For a fixed (p, q), suppose that $f \in \mathfrak{H}^{\otimes(p+q)}$. $\hat{f}$ is the symmetrization of f in the sense of Itô [7]:

$$\tilde{f}(t_1, t_2, \ldots, t_{p+q}) = \frac{1}{p!q!}\sum_{\pi}\sum_{\sigma} f(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(p)}, t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(q)}), \tag{1.1}$$

where π and σ run over all permutations of $(1, 2, \cdots, p)$ and $(p+1, p+2, \cdots, p+q)$ respectively. Denote by $\mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q} = L^2_S(T^p, \mathcal{B}^{\otimes p}, \mu^{\otimes p}) \otimes L^2_S(T^q, \mathcal{B}^{\otimes q}, \mu^{\otimes q})$ the space of square integrable and symmetric functions on T^{p+q} in the above sense. Notice that (1.1) is different to the ordinary symmetrization of f in the theory of real multiple integrals which is given by

$$\hat{f}(t_1, t_2, \cdots, t_{p+q}) = \frac{1}{(p+q)!}\sum_{\pi} f(t_{\pi(1)}, t_{\pi(2)}, \cdots, t_{\pi(p+q)}), \tag{1.2}$$

where π runs over all permutations of $(1, 2, \cdots, p+q)$.

Obviously, we have that (see [7, (5.2)])

$$\|\tilde{f}\| \le \|f\|. \tag{1.3}$$

Definition 1.1[7] (Complex multiple Wiener-Itô integrals) Suppose that $E_1, E_2, \cdots, E_n \subset \mathcal{B}$ is any system of disjoint sets and $e_{i_1\cdots i_p j_1\cdots j_q}$ is a complex-valued function defined for $i_1, i_2, \cdots, i_p$, $j_1, j_2, \cdots, j_q = 1, 2, \cdots, n$ such that $e_{i_1\cdots i_p j_1\cdots j_q} = 0$ unless $i_1, i_2, \cdots, i_p, j_1, j_2, \cdots, j_q$ are all different. Let $\mathcal{S}_{pq}$ denote the set of all functions of the form

$$f(t_1, t_2, \cdots, t_p, s_1, s_2, \cdots, s_q) = \sum e_{i_1\cdots i_p j_1\cdots j_q} \mathbf{1}_{E_{i_1}\times\cdots\times E_{i_p}\times E_{j_1}\times\cdots\times E_{j_q}}, \tag{1.4}$$

where $\mathbf{1}_B(\,\cdot\,)$ is the characteristic function of the set B. The multiple integral of f is defined by

$$I_{p,q}(f) = \sum e_{i_1\cdots i_p j_1\cdots j_q} M(E_{i_1})\cdots M(E_{i_p})\overline{M(E_{j_1})}\cdots\overline{M(E_{j_q})}. \tag{1.5}$$

Clearly, the above integral satisfies that

$$I_{p,q}(f) = I_{p,q}(\tilde{f}), \tag{1.6}$$

$$E[I_{p,q}(f)\overline{I_{p,q}(g)}] = p!q!\langle \tilde{f}, \tilde{g}\rangle, \tag{1.7}$$

$$E[|I_{p,q}(f)|^2] = p!q!\|\tilde{f}\|^2 \le p!q!\,\|f\|^2, \quad \text{(Itô's isometry)} \tag{1.8}$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are the norm and inner product on $\mathfrak{H}^{\otimes(p+q)}$, respectively. Since $\mathcal{S}_{pq}$ is dense in $\mathfrak{H}^{\otimes(p+q)}$, one can extend the integral to any $f\in\mathfrak{H}^{\otimes(p+q)}$ by taking the limit, i.e.,

$$I_{p,q}(f):=\int\cdots\int f\mathrm{d}M(t_1)\cdots\mathrm{d}M(t_p)\overline{\mathrm{d}M(s_1)}\cdots\overline{\mathrm{d}M(s_q)}=\lim_{n\to\infty}I_{p,q}(f_n), \tag{1.9}$$

where $f_n\in\mathcal{S}_{pq}$ such that $f_n\to f$ in $\mathfrak{H}^{\otimes(p+q)}$, and the definition is independent of the choice of the sequence $\{f_n\}$. In addition, (1.6)–(1.8) are still valid to any $f,g\in\mathfrak{H}^{\otimes(p+q)}$. Moreover, the set

$$\mathscr{H}_{p,q}:=\left\{I_{p,q}(f):\ f\in\mathfrak{H}^{\otimes(p+q)}\right\}$$

is called the Wiener-Itô chaos of degree of (p,q) or (p,q)-th Wiener-Itô chaos.

Definition 1.2 (Complex Hermite polynomials) The complex Hermite polynomials $J_{m,n}(z,\rho)$ are given by [7]:

$$\exp\left\{\lambda\bar{z}+\bar{\lambda}z-\rho|\lambda|^2\right\}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\bar{\lambda}^m\lambda^n}{m!n!}J_{m,n}(z,\rho), \tag{1.10}$$

where $\lambda\in\mathbb{C}$. When $\rho=2$, we will often write $J_{m,n}(z)$ rather than $J_{m,n}(z,\rho)$.

Applying [7, Theorem 9] (or see Lemma 2.1 below) and the properties of complex Hermite polynomials (see [7, Theorem 12]), Itô established the relation between complex multiple integrals and complex Hermite polynomials (see [7, Theorem 13.2]): suppose $h_1(t),h_2(t),\cdots,h_l(t)(t_0)$ to be any orthonormal system in $\mathfrak{H}$ and $\alpha_i,\ \beta_j=1,2,\cdots,l$, then

$$\begin{aligned}&\int\cdots\int h_{\alpha_1}(t_1)\cdots h_{\alpha_m}(t_m)\overline{h_{\beta_1}(s_1)}\cdots\overline{h_{\beta_n}(s_n)}\mathrm{d}M(t_1)\cdots\mathrm{d}M(t_m)\overline{\mathrm{d}M(s_1)}\cdots\overline{\mathrm{d}M(s_n)}\\&\quad=\prod_{k=1}^{l}2^{-\frac{m_k+n_k}{2}}J_{m_k,n_k}(\sqrt{2}Z(h_k)),\end{aligned} \tag{1.11}$$

where

$$Z(h_k)=\int h_k(t)\mathrm{d}M(t),\quad k=1,2,\cdots,l, \tag{1.12}$$

and m_k, n_k are the number of k appeared in α_i and β_j respectively.

Remark 1.1 As a result of the above equality and [3, Proposition 2.9], $\mathscr{H}_{p,q}$ is equal to the closed linear subspace of $L^2_{\mathbb{C}}(\mathbf{M})$ generated by the random variables of the type

$$\left\{J_{m,n}(Z(h)):h\in\mathfrak{H},\|h\|_{\mathfrak{H}}=\sqrt{2}\right\}, \tag{1.13}$$

where $Z(h)$ is the same as (1.12). Please refer to Definition 2.7 and Remark 9 of [3] for details.

Notation 1.2 Suppose that $f(t_1,t_2,\cdots,t_p,s_1,s_2,\cdots,s_q)\in\mathfrak{H}^{\odot p}\otimes\mathfrak{H}^{\odot q}$. We call

$$\mathfrak{H}^{\odot q}\otimes\mathfrak{H}^{\odot p}\ni h(t_1,t_2,\cdots,t_q,s_1,s_2,\cdots,s_p):=\bar{f}(s_1,s_2,\cdots,s_p,t_1,t_2,\cdots,t_q)$$

the reversed complex conjugate of function $f(t_1,t_2,\cdots,t_p,s_1,s_2,\cdots,s_q)$.

From Definition 1.1, we can obtain the following lemma easily.

Lemma 1.1 Suppose that $f(t_1,t_2,\cdots,t_p,s_1,s_2,\cdots,s_q)\in\mathfrak{H}^{\odot p}\otimes\mathfrak{H}^{\odot q}$. Let h be the reversed complex conjugate of f, then

$$\overline{I_{p,q}(f)}=I_{q,p}(h). \tag{1.14}$$

We conclude these preliminaries by two propositions, which will be needed throughout the sequel.

Proposition 1.1 (i) Complex multiple Wiener-Itô integrals have all moments satisfying the following hypercontractivity inequality

$$[E\,|I_{p,q}(f)|^r]^{\frac{1}{r}}\le(r-1)^{\frac{p+q}{2}}[E\,|I_{p,q}(f)|^2]^{\frac{1}{2}},\quad r\ge 2, \tag{1.15}$$

where $|\cdot|$ is the absolute value (or modulus) of a complex number.

(ii) If a sequence of distributions of $\{I_{p,q}(f_n)\}_{n\ge1}$ is tight, then

$$\sup_n E\,|I_{p,q}(f)|^r<\infty\quad\text{for every } r>0. \tag{1.16}$$

Proof (i) (1.15) is a consequence of the hypercontractivity of normal Ornstein-Uhlenbeck semigroup[1].

(ii) Along the same line as (ii) of [11, Lemma 2.1] for the case of real multiple integrals, we can show that (1.16) holds.

2 The Product Formula for Complex Multiple Wiener-Itô Integrals

Notation 2.1 For two symmetric functions $f\in\mathfrak{H}^{\odot p_1}\otimes\mathfrak{H}^{\odot q_1}$, $g\in\mathfrak{H}^{\odot p_2}\otimes\mathfrak{H}^{\odot q_2}$ and $i\le p_1\wedge q_2$, $j\le q_1\wedge p_2$, the contraction of (i,j) indices of the two functions is given by

$$\begin{aligned}
&f\otimes_{i,j}g(t_1,\cdots,t_{p_1+p_2-i-j};s_1,\cdots,s_{q_1+q_2-i-j})\\
&\quad=\int_{A^{i+j}}\mu^{i+j}(\mathrm{d}u_1\cdots\mathrm{d}u_i\mathrm{d}v_1\cdots\mathrm{d}v_j)\,f(t_1,\cdots,t_{p_1-i},u_1,\cdots,u_i;s_1\cdots,s_{q_1-j},v_1\cdots,v_j)\\
&\qquad\times g(t_{p_1-i+1},\cdots,t_{p_1-i+p_2-j},v_1\cdots,v_j;s_{q_1-j+1}\cdots,s_{q_1-j+q_2-i},u_1\cdots,u_i);
\end{aligned}\tag{2.1}$$

by convention, $f\otimes_{0,0}g=f\otimes g$ denotes the tensor product of f and g. We write $f\tilde{\otimes}_{p,q}g$ for the symmetrization of $f\otimes_{p,q}g$. In what follows, we use the convention $f\otimes_{i,j}g=0$ if $i>p_1\wedge q_2$ or $j>q_1\wedge p_2$.

The following lemma is the starting point of the relationship (1.11) and the product formula for complex multiple Wiener-Itô integrals.

Lemma 2.1[7, Theorem 9] Let $f\in\mathfrak{H}^{\odot p}\otimes\mathfrak{H}^{\odot q}$ be a symmetric function and let $g\in\mathfrak{H}$. Then

$$I_{p,q}(f)I_{1,0}(g)=I_{p+1,q}(f\otimes g)+qI_{p,q-1}(f\otimes_{0,1}g), \tag{2.2}$$

$$I_{p,q}(f)I_{0,1}(g)=I_{p,q+1}(f\otimes g)+pI_{p-1,q}(f\otimes_{1,0}g). \tag{2.3}$$

Theorem 2.1 (Product formula) For two symmetric functions $f \in \mathfrak{H}^{\odot a} \otimes \mathfrak{H}^{\odot b}$, $g \in \mathfrak{H}^{\odot c} \otimes \mathfrak{H}^{\odot d}$, the product formula for complex multiple Wiener-Itô integrals is given by

$$I_{a,b}(f)I_{c,d}(g) = \sum_{i=0}^{a\wedge d}\sum_{j=0}^{b\wedge c}\binom{a}{i}\binom{d}{i}\binom{b}{j}\binom{c}{j} i!j!\, I_{a+c-i-j,b+d-i-j}(f \otimes_{i,j} g), \tag{2.4}$$

where $a, b, c, d \in \mathbb{N}$.

Remark 2.1 Hoshino et al. obtained a more general product formula for complex Wiener-Itô integrals where f, g can be non-symmetrized functions (see [9, Theorem A.1]). When f, g are restricted to symmetric functions, their product formula is coincident with ours and thus gives a very clear explanation of the combination numbers in (2.4).

Proof From Remark 1.1, we only need to show that (2.4) holds for $I_{a,b}(f) = J_{a,b}(Z(h_1))$ and $I_{c,d}(g) = J_{c,d}(Z(h_2))$ with $h_1, h_2 \in \mathfrak{H}$ such that $\|h_1\| = \|h_2\| = \sqrt{2}$ by a density argument. That is to say,

$$f = h_1^{\otimes a} \otimes \bar{h}_1^{\otimes b}, \quad g = h_2^{\otimes c} \otimes \bar{h}_2^{\otimes d}.$$

By the decomposition theorem of Hilbert spaces, we may as well assume that $h_1 = h_2$ or $\langle h_1, h_2\rangle_{\mathfrak{H}} = 0$.

It follows from the generating function of complex Hermite polynomials[2, 7] that

$$\begin{aligned}
&\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{\bar{\lambda}^a\lambda^b}{a!b!}J_{a,b}(z)\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\frac{\bar{\mu}^c\mu^d}{m!n!}J_{c,d}(z)\\
&\quad= \exp\left\{\lambda\bar{z} + \bar{\lambda}z - 2|\lambda|^2\right\}\exp\left\{\mu\bar{z} + \bar{\mu}z - 2|\lambda|^2\right\}\\
&\quad= \exp\left\{(\lambda+\mu)\bar{z} + \overline{(\lambda+\mu)}z - 2|\lambda+\mu|^2\right\}\exp\left\{2(\bar{\lambda}\mu + \lambda\bar{\mu})\right\}\\
&\quad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\overline{(\lambda+\mu)^m}(\lambda+\mu)^n}{m!n!}J_{m,n}(z)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^{i+j}(\bar{\lambda}\mu)^i(\lambda\bar{\mu})^j}{i!j!},
\end{aligned}$$

where $z, \lambda, \mu \in \mathbb{C}$. Expanding $(\lambda+\mu)^n$ and comparing coefficients immediately yield

$$J_{a,b}(z)J_{c,d}(z) = \sum_{i=0}^{a\wedge d}\sum_{j=0}^{b\wedge c}\binom{a}{i}\binom{d}{i}\binom{b}{j}\binom{c}{j} i!j!2^{i+j}\, J_{a+c-i-j,b+d-i-j}(z). \tag{2.5}$$

When $h_1 = h_2$, it follows from the relationship (1.11) that (2.4) is exactly (2.5).

When $\langle h_1, h_2\rangle_{\mathfrak{H}} = 0$, we have

$$f \otimes_{i,j} g = \begin{cases} 0, & i+j > 0,\\ (h_1^{\otimes a} \otimes h_2^{\otimes c}) \otimes (\bar{h}_1^{\otimes b} \otimes \bar{h}_2^{\otimes d}), & i = j = 0.\end{cases}$$

Thus, (2.4) is degenerated to the relationship (1.11) in this case.

Remark 2.2 There are another two approaches to show Theorem 2.1. One is by induction over the indices c and d (see [12, Proposition 1.1.3]) using Lemma 2.1, which involves some tedious combinatorial calculations. The other is by Malliavin calculus (see [10, Theorem 2.7.10]) if we exploit the framework of complex Malliavin operators.

The following product formulas, which is a direct corollary of Lemma 1.1 and Theorem 2.1, will be used later.

Corollary 2.1

$$I_{a,b}(f)\overline{I_{c,d}(g)} = \sum_{i=0}^{a\wedge c}\sum_{j=0}^{b\wedge d}\binom{a}{i}\binom{c}{i}\binom{b}{j}\binom{d}{j} i!j!\, I_{a+d-i-j,b+c-i-j}(f\otimes_{i,j} h),$$

where h is the reversed complex conjugate of g (see Notation 1.2), and

$$\begin{aligned} & f\otimes_{i,j} h(t_1,\cdots,t_{a+d-i-j};s_1,\cdots,s_{b+c-i-j}) \\ &\quad = \int_{A^{i+j}} \mu^{i+j}(\mathrm{d}u_1\cdots\mathrm{d}u_i\mathrm{d}v_1\cdots\mathrm{d}v_j)\, f(t_1,\cdots,t_{a-i},u_1,\cdots,u_i;s_1,\cdots,s_{b-j},v_1\cdots,v_j) \\ &\quad\quad \times \bar{g}(s_{b-j+1},\cdots,s_{b-j+c-i},u_1,\cdots,u_i;t_{a-i+1},\cdots,t_{a-i+d-j},v_1,\cdots,v_j). \end{aligned} \tag{2.6}$$

3 The Independence of Complex Multiple Wiener-Itô Integrals

Lemma 3.1 For two symmetric functions $f\in\mathfrak{H}^{\odot a}\otimes\mathfrak{H}^{\odot b}$, $g\in\mathfrak{H}^{\odot c}\otimes\mathfrak{H}^{\odot d}$, let $F=I_{a,b}(f)$, $G=I_{c,d}(g)$ and h be the reversed complex conjugate of g. Then

$$\begin{aligned}\mathrm{Cov}(|F|^2,\,|G|^2) &= \sum_{i+j>0}\binom{a}{i}\binom{c}{i}\binom{b}{j}\binom{d}{j} a!b!c!d!\,\|f\otimes_{i,j} g\|^2_{\mathfrak{H}^{\otimes(m-2(i+j))}} \\ &\quad + \sum_{r\geq 1}(a+d-r)!(b+c-r)!\,\|\phi_r\|^2_{\mathfrak{H}^{\otimes(m-2r)}}\,,\end{aligned}$$

where

$$\phi_r = \sum_{i+j=r}\binom{a}{i}\binom{c}{i}\binom{b}{j}\binom{d}{j} i!j! f\tilde{\otimes}_{i,j} h. \tag{3.1}$$

As a consequence, the squares of the absolute values of complex multiple Wiener-Itô integrals are non-negatively correlated.

Proof We divide the proof into three steps. Let $m=a+b+c+d$.

First, it follows from Corollary 2.1, the orthogonal property and Itô's isometry of multiple Wiener-Itô integrals that

$$E[|F\bar{G}|^2] = \sum_{r\geq 0}(a+d-r)!(b+c-r)!\,\|\phi_r\|^2_{\mathfrak{H}^{\otimes(m-2r)}}\,.$$

Second, we claim that

$$(a+d)!(b+c)!\left\|f\tilde{\otimes}h\right\|^2 = \sum_{i=0}^{a\wedge d}\sum_{j=0}^{b\wedge c}\binom{a}{i}\binom{d}{i}\binom{b}{j}\binom{c}{j} a!b!c!d!\,\|f\otimes_{i,j} g\|^2_{\mathfrak{H}^{\otimes(m-2(i+j))}}\,.$$

Let π (σ resp.) be a permutation of the set $\{1,2,\cdots,a+d\}$ ($\{1,2,\cdots,b+c\}$ resp.). Denote by π_0, σ_0 the identity permutations. We write $\pi\sim_i\pi_0$ ($\sigma\sim_i\sigma_0$ resp.) if the set $\{\pi(1),\pi(2),\cdots,\pi(a)\}\cap\{1,2,\cdots,a\}$ ($\{\sigma(1),\sigma(2),\cdots,\pi(b)\}\cap\{1,2,\cdots,b\}$ resp.) contains exactly i elements[13]. Then we have

$$\begin{aligned}
&(a+d)!(b+c)!\left\|f\tilde{\otimes}h\right\|^2=(a+d)!(b+c)!\langle f\otimes h,\, f\tilde{\otimes}h\rangle\\
&\quad=\sum_{\pi}\sum_{\sigma}\int_{A^m}\mathrm{d}\mu^m f(t_1,\cdots,t_a,s_1\cdots,s_b)\bar{g}(s_{b+1},\cdots,s_{b+c},t_{a+1},\cdots,t_{a+d})\\
&\quad\quad\times\bar{f}(t_{\pi(1)},\cdots,t_{\pi(a)},s_{\sigma(1)}\cdots,s_{\sigma(b)})g(s_{\sigma(b+1)},\cdots,s_{\sigma(b+c)},t_{\pi(a+1)},\cdots,t_{\pi(a+d)})\\
&\quad=\sum_{i=0}^{a\wedge d}\sum_{j=0}^{b\wedge c}\sum_{\pi\sim_{a-i}\pi_0}\sum_{\sigma\sim_{b-j}\sigma_0}\|f\otimes_{i,j}g\|^2_{\mathfrak{H}^{\otimes(m-2(i+j))}}\\
&\quad=\sum_{i=0}^{a\wedge d}\sum_{j=0}^{b\wedge c}\binom{a}{i}\binom{d}{i}\binom{b}{j}\binom{c}{j}a!b!c!d!\,\|f\otimes_{i,j}g\|^2_{\mathfrak{H}^{\otimes(m-2(i+j))}}.
\end{aligned}$$

Third, note that when $i=j=0$,

$$\|f\otimes_{i,j}g\|^2_{\mathfrak{H}^{\otimes(m-2(i+j))}}=\|f\|^2\|g\|^2.$$

Itô's isometry implies that

$$E[|F|^2]E[|G|^2]=a!b!c!d!\,\|f\|^2\|g\|^2.$$

Thus,

$$\begin{aligned}
\mathrm{Cov}(|F|^2,\,|G|^2)&=E[|F\bar{G}|^2]-E[|F|^2]E[|G|^2]\\
&=\sum_{i+j>0}\binom{a}{i}\binom{c}{i}\binom{b}{j}\binom{d}{j}a!b!c!d!\,\|f\otimes_{i,j}g\|^2_{\mathfrak{H}^{\otimes(m-2(i+j))}}\\
&\quad+\sum_{r\geq 1}(a+d-r)!(b+c-r)!\,\|\phi_r\|^2_{\mathfrak{H}^{\otimes(m-2r)}}.
\end{aligned}$$

Theorem 3.1 (Üstünel-Zakai independence criterion) For two symmetric functions $f\in\mathfrak{H}^{\odot a}\otimes\mathfrak{H}^{\odot b}$, $g\in\mathfrak{H}^{\odot c}\otimes\mathfrak{H}^{\odot d}$ with $a+b\geq 1$, $c+d\geq 1$, the following conditions are equivalent:

(i) $I_{a,b}(f)$ and $I_{c,d}(g)$ are independent random variables;

(ii) $f\otimes_{1,0}g=0$, $f\otimes_{0,1}g=0$, $f\otimes_{1,0}h=0$ and $f\otimes_{0,1}h=0$ a.e. μ^{m-2}, where $m=a+b+c+d$ and h is the reversed complex conjugate of g.

Proof (i)$\Rightarrow$ (ii): Denote $F=I_{a,b}(f)$, $G=I_{c,d}(g)$. It follows from Proposition 1.1 (1) that inside a fixed Wiener chaos (i.e., for the fixed (a,b)), all the L^q-norms ($q>1$) are equivalent. Thus $\mathrm{Cov}(|F|^2,|G|^2)$ is finite. If (i) is satisfied then $\mathrm{Cov}(|F|^2,|G|^2)=0$. It follows from Lemma 3.1 that $f\otimes_{1,0}g=0$ and $f\otimes_{0,1}g=0$. Note that $\bar{G}=I_{d,c}(h)$ and F, $\bar{G}$ are also independent random variables. Thus we also have $f\otimes_{1,0}h=0$ and $f\otimes_{0,1}h=0$.

(ii)$\Rightarrow$ (i): We divide the proof into three steps along the same line as the proof for real multiple integrals[8].

First, let $\mathcal{H}_f$, $\mathcal{G}_f$ respectively denote the Hilbert subspace in $\mathfrak{H}$ spanned by all functions

$$t \mapsto \int_{A^{a+b-1}} f(t, x_1, \cdots, x_{a+b-1}) h(x_1, \cdots, x_{a+b-1}) \mu^{a+b-1}(\mathrm{d}x_1 \cdots \mathrm{d}x_{a+b-1}),$$
$$t \mapsto \int_{A^{a+b-1}} f(x_1, \cdots, x_{a+b-1}, t) h(x_1, \cdots, x_{a+b-1}) \mu^{a+b-1}(\mathrm{d}x_1 \cdots \mathrm{d}x_{a+b-1}),$$

where $t \in A$ and $h \in \mathfrak{H}^{\otimes(a+b-1)}$. Similarly we define $\mathcal{H}_g$, $\mathcal{G}_g$ in terms of g. Denote by $\overline{\mathcal{G}_f}$ the complex conjugate of $\mathcal{G}_f$. We claim that Condition (ii) implies that $\{\mathcal{H}_f, \overline{\mathcal{G}_f}\}$ and $\{\mathcal{H}_g, \overline{\mathcal{G}_g}\}$ are orthogonal. In fact, $f \otimes_{1,0} g = 0$, $f \otimes_{0,1} g = 0$, $f \otimes_{1,0} h = 0$ and $f \otimes_{0,1} h = 0$ respectively imply that $\mathcal{H}_f \perp \overline{\mathcal{G}_g}$, $\overline{\mathcal{G}_f} \perp \mathcal{H}_g$, $\mathcal{H}_f \perp \mathcal{H}_g$ and $\overline{\mathcal{G}_f} \perp \overline{\mathcal{G}_g}$ using Fubini Theorem. For example, letting $h(x) \in \mathfrak{H}^{\otimes(a+b-1)}$, $l(y) \in \mathfrak{H}^{\otimes(c+d-1)}$ and $m = a+b+c+d$, we have

$$\begin{aligned} &\int_A \mu(\mathrm{d}t) \int_{A^{a+b-1}} f(t,x) h(x) \mu^{a+b-1}(\mathrm{d}x) \int_{A^{c+d-1}} g(y,t) l(y) \mu^{c+d-1}(\mathrm{d}y) \\ &\quad= \int_{A^{m-2}} \mu^{m-2}(\mathrm{d}x\mathrm{d}y) h(x) l(y) \int_A f(t,x) g(y,t) \mu(\mathrm{d}t) \\ &\quad= \int_{A^{m-2}} h(x) l(y) f \otimes_{1,0} g \, \mu^{m-2}(\mathrm{d}x\mathrm{d}y) = 0, \end{aligned}$$

i.e., $\mathcal{H}_f$ and $\overline{\mathcal{G}_g}$ are orthogonal.

Second, let $\{\varphi_n\}$ ($\{\psi_n\}$ resp.) be an orthonormal basis for the Hilbert subspace in $\mathfrak{H}$ spanned by $\{\mathcal{H}_f, \overline{\mathcal{G}_f}\}$ ($\{\mathcal{H}_g, \overline{\mathcal{G}_g}\}$ resp.). Since the tensor products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_a} \otimes \bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_b}$ form an orthonormal basis in $\mathcal{H}_f^{\otimes a} \otimes \overline{\mathcal{G}_f}^{\otimes b}$, it follows from monotonic class theorem[4–5] that f (and g) can be decomposed as

$$f = \sum e_{i_1 \cdots i_a j_1 \cdots j_b} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_a} \otimes \bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_b},$$
$$g = \sum l_{i_1 \cdots i_c j_1 \cdots j_d} \psi_{i_1} \otimes \cdots \otimes \psi_{i_c} \otimes \bar{\psi}_{j_1} \otimes \cdots \otimes \bar{\psi}_{j_d}.$$

Third, we claim that F, G are independent. In fact, we write

$$\xi_i = \int_A \varphi_i(a) M(\mathrm{d}a) \quad \text{and} \quad \eta_j = \int_A \psi_j M(\mathrm{d}a)$$

for all i and j. Then the Cramér-Wold theorem implies that the entire sequences $\{\xi_i\}$ and $\{\eta_j\}$ are independent[8]. It follows from (1.11) that $F = I_{a,b}(f)$ ($G = I_{c,d}(g)$ resp.) can be expanded into polynomials in $\xi_1, \xi_2, \cdots$ ($\eta_1, \eta_2, \cdots$ resp.). Thus F, G are independent.

Remark 3.1 Similar to real multiple integrals[14], Condition (i) of Theorem 3.1 is also equivalent to

(iii) $\mathrm{Cov}(|F|^2, |G|^2) = 0$, i.e., $|F|^2$, $|G|^2$ are irrelevant,

which can be observed from the above proof.

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复多重维纳—伊藤积分的乘法公式和独立性

陈 勇

(湖南科技大学数学与计算科学学院, 湘潭, 湖南, 411201)

摘要: 本文给出复多重维纳—伊藤积分的一个乘法公式. 作为应用, 本文证明了复多重维纳—伊藤积分独立性的 Üstünel-Zakai 准则.

关键词: 复多重维纳—伊藤积分; 乘法公式; Üstünel-Zakai 准则