第十四章　线性动态电路的复频域分析

本章重点：

1. 拉普拉斯变换的原理和性质
2. 运算法分析线性电路
3. 用网络函数求零状态响应

Pierre Simon Laplace
(1749-1827),

拉普拉斯

a French astronomer and mathematician, first presented the transform that bears his name — Laplace transform — and its applications to differential equations in 1779.

§14-1 拉普拉斯变换的定义（Laplace Transform)

通过积分变换将时域函数变换为频域函数，从而将时域的微分方程化为频域函数的代数方程。

复频域　象函数 $F(s)= \mathcal{L}\,[f(t)]$ 原函数 时域

复频率 $s = \sigma + j\omega$

拉氏变换：
- 双边 $F(s)=\int_{-\infty}^{\infty} f(t)\cdot e^{-st}\mathrm{d}t$
- 单边 $F(s)=\int_{0_-}^{\infty} f(t)\cdot e^{-st}\mathrm{d}t$

拉氏反变换 $f(t)=\dfrac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\cdot e^{st}\mathrm{d}s$

拉氏变换对

例13-1 求以下函数的象函数

1、$\varepsilon(t)$　2、$\delta(t)$　3、$e^{-\alpha t}$

解：1、
$$F(s)=\int_{0_-}^{\infty} f(t)\cdot e^{-st}\mathrm{d}t =\int_{0_-}^{\infty} \varepsilon(t)\cdot e^{-st}\mathrm{d}t =\int_{0_-}^{\infty} e^{-st}\mathrm{d}t = -\frac{1}{s}\cdot e^{-st}\Big|_{0_-}^{\infty} = \frac{1}{s}$$

例13-1 求以下函数的象函数

1、$\varepsilon(t)$　2、$\delta(t)$　3、$e^{-\alpha t}$

2、
$$F(s)=\int_{0_-}^{\infty} f(t)\cdot e^{-st}\mathrm{d}t =\int_{0_-}^{\infty} \delta(t)\cdot e^{-st}\mathrm{d}t =\int_{0_-}^{\infty} \delta(t)\cdot e^{-s\times 0}\mathrm{d}t =\int_{0_-}^{\infty} \delta(t)\mathrm{d}t = 1$$

例13-1 求以下函数的象函数

1、$\varepsilon(t)$ 2、$\delta(t)$ 3、e^{-at}

3、$F(s)=\int_{0_-}^{\infty} f(t)\cdot e^{-st}dt$

$=\int_{0_-}^{\infty} e^{-at}\cdot e^{-st}dt$

$=\int_{0_-}^{\infty} e^{-(s+a)t}dt$

$=\dfrac{1}{s+a}$

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$\mathcal{L}[f(t)]=F(s)$

$\mathcal{L}[f(t)\varepsilon(t)]=?$

$\mathcal{L}[f(t)]=\mathcal{L}[f(t)\varepsilon(t)]$

除冲激函数外的任意函数

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§14-2 拉普拉斯变换的基本性质

1、线性性质

$\mathcal{L}[f_1(t)]=F_1(s)$, $\mathcal{L}[f_2(t)]=F_2(s)$

则：$\mathcal{L}[A_1f_1(t)+A_2f_2(t)]=A_1F_1(s)+A_2F_2(s)$

补例1 求 $(2-e^{-t}+e^{-2t})\varepsilon(t)$ 的象函数

$$F(s)=\frac{2}{s}-\frac{1}{s+1}+\frac{1}{s+2}$$

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2、时域微分性质 $\mathcal{L}[f(t)]=F(s)$

则：$\mathcal{L}\left[\dfrac{df(t)}{dt}\right]=sF(s)-f(0_-)$

重复推导可得

$$\mathcal{L}[f^{(n)}(t)]=s^nF(s)-\sum_{j=0}^{n-1}s^{n-1-j}\left.\frac{d^jf(t)}{dt^j}\right|_{t=0_-}$$

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3、积分性质 $\mathcal{L}[f(t)]=F(s)$

则：$\mathcal{L}\left[\int_{0_-}^{t} f(t)dt\right]=\dfrac{F(s)}{s}$

4、延迟(时域位移)性质

$\mathcal{L}[f(t)\varepsilon(t)]=F(s)$

则：$\mathcal{L}[f(t-t_0)\varepsilon(t-t_0)]=e^{-st_0}F(s)$

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补例2 求以下函数的象函数

1. $k\varepsilon(t-t_0)$ $\qquad \mathcal{L}[\varepsilon(t)]=\dfrac{1}{s}$

$\mathcal{L}[k\varepsilon(t-t_0)]=\dfrac{k}{s}\cdot e^{-st_0}$

2. $(t-2)\varepsilon(t-2)$ $\qquad \mathcal{L}[t\varepsilon(t)]=\dfrac{1}{s^2}$

$\mathcal{L}[(t-2)\varepsilon(t-2)]=\dfrac{1}{s^2}\cdot e^{-2s}$

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补例2 求以下函数的象函数

$3.\ (t-2)\varepsilon(t) = t\varepsilon(t) - 2\varepsilon(t)$

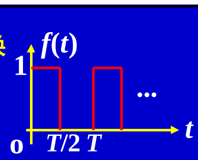
$$\mathcal{L}\,[(t-2)\varepsilon(t)] = \frac{1}{s^2} - \frac{2}{s}$$

$4.\ t\varepsilon(t-2)$

$$= (t-2)\varepsilon(t-2) + 2\varepsilon(t-2)$$

$$\mathcal{L}\,[t\varepsilon(t-2)] = \frac{e^{-2s}}{s^2} + \frac{2}{s}\cdot e^{-2s}$$

补例3 求周期函数的拉氏变换



解 设$f_1(t)$为一个周期的函数

$$\mathcal{L}[f_1(t)] = F_1(s)$$

$$\because f(t) = f_1(t) + f_1(t-T)\varepsilon(t-T) + f_1(t-2T)\varepsilon(t-2T) + \cdots$$

$$\mathcal{L}[f(t)] = F_1(s) + e^{-sT}F_1(s) + e^{-2sT}F_1(s) + \cdots$$

$$= F_1(s)[e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots]$$

$$= \frac{1}{1-e^{-sT}}F_1(s)$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}F_1(s)$$

$$f_1(t) = \varepsilon(t) - \varepsilon(t - \frac{T}{2})$$

$$\longrightarrow F_1(s) = (\frac{1}{s} - \frac{1}{s}e^{-sT/2})$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}(\frac{1}{s} - \frac{1}{s}e^{-sT/2})$$

$$= \frac{1}{s}(\frac{1}{1+e^{-sT/2}})$$

5、复频域微分性质 $\mathcal{L}\,[f(t)] = F(s)$

则：$\mathcal{L}\,[t\cdot f(t)] = -\dfrac{dF(s)}{ds}$

补例4 求 $t^n\varepsilon(t)$ 的象函数

$$\mathcal{L}\,[\varepsilon(t)] = \frac{1}{s} \qquad \mathcal{L}\,[t\cdot\varepsilon(t)] = -\frac{d}{ds}\mathcal{L}\,[\varepsilon(t)] = \frac{1}{s^2}$$

$$\mathcal{L}\,[t^2\cdot\varepsilon(t)] = -\frac{d}{ds}\mathcal{L}\,[t\varepsilon(t)] = \frac{2}{s^3} \quad \cdots$$

$$\mathcal{L}\,[t^n\cdot\varepsilon(t)] = \frac{n!}{s^{n+1}}$$

6、复频域位移性质 $\mathcal{L}\,[f(t)] = F(s)$

则：$\mathcal{L}\,[e^{-\alpha t}\cdot f(t)] = F(s+\alpha)$

补例5 求 $e^{-\alpha t}t^n\varepsilon(t)$ 的象函数

$$\mathcal{L}\,[t^n\cdot\varepsilon(t)] = \frac{n!}{s^{n+1}}$$

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$$\mathcal{L}\,[e^{-\alpha t}t^n\cdot\varepsilon(t)] = \frac{n!}{(s+\alpha)^{n+1}}$$

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§14-3 拉普拉斯反变换的部分分式展开

numerator

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}$$

denominator

$n > m$ 真分式

电路分析中，一般 $n \geq m$

$n = m$ 假分式

部分分式展开

一、$D(s)=0$ 只有单根

n个单根：$p_1, p_2, \cdots, p_n$

$$D(s) = b_n(s-p_1)(s-p_2)\cdots(s-p_n)$$

$$F(s) = \frac{N(s)}{D(s)} = \sum_{i=1}^{n} \frac{k_i}{s-p_i}$$

$$F(s)(s-p_i) = k_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{k_j(s-p_i)}{s-p_j}$$

$$k_i = F(s)(s-p_i)\Big|_{s=p_i}$$

$$F(s) = \frac{N(s)}{D(s)} = \sum_{i=1}^{n} \frac{k_i}{s-p_i}$$

洛必达法则：$k_i = \left.\dfrac{N(s)}{D'(s)}\right|_{s=p_i}$

$$\mathcal{L}\left[e^{-at}\right] = \frac{1}{s+a}$$

原函数为：$f(t) = \sum_{i=1}^{n} k_i e^{p_i t}\varepsilon(t)$

补例1 求以下函数的原函数

$$F(s) = \frac{7s+2}{s^2+3s+2} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2}$$

解：$D(s) = s^2+3s+2 = (s+1)(s+2)$

$D(s)=0$ ， $p_1=-1$ ， $p_2=-2$

$$k_1 = F(s)(s-p_1)\Big|_{s=p_1} = \left.\frac{7s+2}{s+2}\right|_{s=-1} = -5$$

补例1 求以下函数的原函数

$$F(s) = \frac{7s+2}{s^2+3s+2} = \frac{-5}{s+1} + \frac{12}{s+2}$$

$$k_2 = F(s)(s-p_2)\Big|_{s=p_2} = 12$$

$$f(t) = \sum_{i=1}^{n} k_i e^{p_i t}\varepsilon(t)$$

$$= -5e^{-t}\varepsilon(t) + 12e^{-2t}\varepsilon(t)$$

二、$D(s)=0$ 有共轭复根

$p_1 = -\alpha + j\omega \quad p_2 = -\alpha - j\omega$

$$k_1 = (s+\alpha-j\omega)F(s)\Big|_{s=-\alpha+j\omega} = K\angle\theta$$

$$k_2 = (s+\alpha+j\omega)F(s)\Big|_{s=-\alpha-j\omega} = K\angle-\theta$$

$$f(t) = k_1 e^{p_1 t}\varepsilon(t) + k_2 e^{p_2 t}\varepsilon(t)$$

$$= \left(Ke^{j\theta}e^{(-\alpha+j\omega)t} + Ke^{-j\theta}e^{(-\alpha-j\omega)t}\right)\varepsilon(t)$$

$$= 2Ke^{-\alpha t}\cos(\omega t+\theta)\varepsilon(t)V \quad ★$$

补例2 求原函数 $F(s) = \dfrac{2s+2}{s^2+6s+13}$

解：$D(s) = s^2+6s+13 = (s+3)^2 - (j2)^2$

$D(s)=0$， $p_{1,2} = -3 \pm j2 = -\alpha \pm j\omega$

$$k_1 = F(s)(s-p_1)\Big|_{s=p_1} = \left.\frac{2s+2}{s+3+j2}\right|_{s=-3+j2}$$

$$= \frac{-4+j4}{j4} = 1+j1 = \sqrt{2}e^{j45°} = Ke^{j\theta}$$

$$f(t) = 2\sqrt{2}e^{-3t}\cos(2t+45°)\varepsilon(t)$$

补例2 求原函数 $F(s)=\dfrac{2s+2}{s^2+6s+13}$

$$=\frac{2s+2}{(s+3)^2+2^2}=\frac{2(s+3)-2\times 2}{(s+3)^2+2^2}$$

$$\begin{cases} \mathcal{L}\,[e^{-\alpha t}\sin\omega t]=\dfrac{\omega}{(s+\alpha)^2+\omega^2} \\ \mathcal{L}\,[e^{-\alpha t}\cos\omega t]=\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2} \end{cases}$$

$$f(t)=2e^{-3t}[\cos(2t)-\sin(2t)]\varepsilon(t)$$

三、$D(s)=0$ 有q阶重根p_1

计算方法同单根

$$F(s)=\sum_{i=2}^{n-q+1}\frac{k_i}{s-p_i}+\frac{k_{1q}}{(s-p_1)^q}+\frac{k_{1(q-1)}}{(s-p_1)^{q-1}}+\dots+\frac{k_{11}}{s-p_1}$$

$$k_{1j}=\frac{1}{(q-j)!}\frac{d^{(q-j)}}{ds^{(q-j)}}[F(s)(s-p_1)^q]\Big|_{s=p_1}$$

$$f(t)=\sum_{j=1}^{q}\frac{k_{1j}}{(j-1)!}t^{j-1}e^{p_1t}\varepsilon(t)+\sum_{i=2}^{n-q+1}k_ie^{p_it}\varepsilon(t)$$

补例3 求以下函数的原函数

$$F(s)=\frac{3s+2}{(s+2)(s+1)^2}=\frac{k_{12}}{(s+1)^2}+\frac{k_{11}}{s+1}+\frac{k_2}{s+2}$$

解： $D(s)=0$ ， $p_1=-1$ ，$p_2=-2$

$$k_2=F(s)(s-p_2)\big|_{s=p_2}$$

$$=\frac{3s+2}{(s+1)^2}\Big|_{s=-2}=-4$$

$$k_{1j}=\frac{1}{(q-j)!}\frac{d^{(q-j)}}{ds^{(q-j)}}[F(s)(s-p_1)^q]\Big|_{s=p_1}$$

$q=2, j=2$

$$k_{12}=F(s)(s-p_1)^2\big|_{S=p_1}$$

$$=\frac{3s+2}{s+2}\Big|_{s=-1}=-1$$

$$k_{1j}=\frac{1}{(q-j)!}\frac{d^{(q-j)}}{ds^{(q-j)}}[F(s)(s-p_1)^q]\Big|_{s=p_1}$$

$j=1$

$$k_{11}=\frac{1}{(2-1)!}\frac{d}{ds}[F(s)(s-p_1)^2]\Big|_{s=p_1}$$

$$=\frac{d}{ds}\left[\frac{3s+2}{s+2}\right]\Big|_{s=-1}=\frac{3\times(s+2)-(3s+2)\times 1}{(s+2)^2}\Big|_{s=-1}$$

$$\left(\frac{a}{b}\right)'=\frac{a'b-ab'}{b^2}$$

$$=\frac{4}{(s+2)^2}\Big|_{s=-1}=4$$

补例3 求以下函数的原函数

$$F(s)=\frac{3s+2}{(s+2)(s+1)^2}=\frac{k_{12}}{(s+1)^2}+\frac{k_{11}}{s+1}+\frac{k_2}{s+2}$$

$$F(s)=\frac{-1}{(s+1)^2}+\frac{4}{s+1}+\frac{-4}{s+2}$$

$$f(t)=(-te^{-t}+4e^{-t}-4e^{-2t})\varepsilon(t)$$

四、$F(s)$为假分式 $\mathcal{L}[\delta^{(n)}(t)]=s^n$

补例4 求以下函数的原函数

$$F(s)=\frac{3s+2}{s+1}=3-\frac{1}{s+1}$$

$$f(t)=3\delta(t)-e^{-t}\varepsilon(t)$$

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§14-4 运算电路

一、KCL，KVL的复频域形式

$$\sum i=0 \qquad \sum u=0 \quad \Rightarrow \quad \sum I(s)=0 \qquad \sum U(s)=0$$

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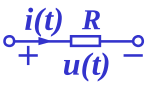
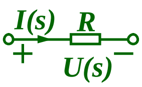
二、电路元件的复频域形式

1、R



$u=Ri$

$\Downarrow$



$U(s)=RI(s)$

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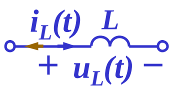
2、L

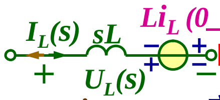


$$u_L=L\frac{di_L}{dt}$$

$$U_L(s)=L\cdot[sI_L(s)-i_L(0\_)]$$
$$=sLI_L(s)-Li_L(0\_)$$

$$u_L=-L\frac{di_L}{dt}$$

$Li_L(0\_)$ 大小



$$U_L(s)=-L\cdot[sI_L(s)-i_L(0\_)]$$
$$=-sLI_L(s)+Li_L(0\_)$$

方向：和电流反向

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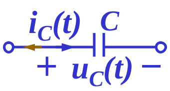
3、C



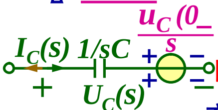
$$i_C=C\frac{du_C}{dt}$$

$$I_C(s)=sCU_C(s)-Cu_C(0\_)$$

$$U_C(s)=\frac{1}{sC}I_C(s)+\frac{u_C(0\_)}{s}$$

$$i_C=-C\frac{du_C}{dt}$$

$\frac{u_C(0\_)}{s}$ 大小



$$U_C(s)=-\frac{1}{sC}I_C(s)+\frac{u_C(0\_)}{s}$$

方向：和电压同向

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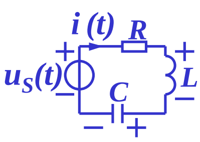
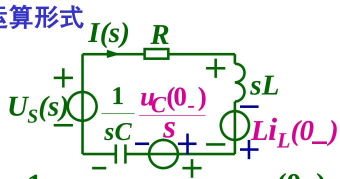
三、欧姆定律的运算形式





$$U_S(s)=(R+sL+\frac{1}{sC})I(s)-Li_L(0\_)+\frac{u_C(0\_)}{s}$$

若初值为0：

$$\begin{cases}U(s)=(R+sL+\dfrac{1}{sC})I(s)=Z(s)I(s)\\ I(s)=Y(s)U(s)\end{cases}$$

运算阻抗

运算导纳

$$Y(s)=\frac{1}{Z(s)}$$

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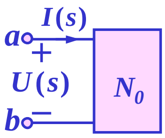
运算阻抗

$$Z(s) \overset{def}{=} \frac{U(s)}{I(s)}$$

$$Y(s) \overset{def}{=} \frac{I(s)}{U(s)}$$

运算导纳



无源网络

	运算阻抗	运算导纳
R	$Z_R(s)=R$	$Y_R(s)=G$
L	$Z_L(s)=sL$	$Y_L(s)=\frac{1}{sL}$
C	$Z_C(s)=\frac{1}{sC}$	$Y_C(s)=sC$