第三节 基变换与坐标变换

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一、基变换公式与过渡矩阵

问题：在 n 维线性空间 V 中，任意 n 个线性无关的向量都可以作为 V 的一组基．对于不同的基，同一个向量的坐标是不同的．

那么，同一个向量在不同的基下的坐标有什么关系呢？换句话说，随着基的改变，向量的坐标如何改变呢？

设$\alpha_1,\alpha_2,\cdots,\alpha_n$及$\beta_1,\beta_2,\cdots,\beta_n$是线性空间$V_n$的两个基,且有

$$\begin{cases} \beta_1 = p_{11}\alpha_1 + p_{21}\alpha_2 + \cdots + p_{n1}\alpha_n \\ \beta_2 = p_{12}\alpha_1 + p_{22}\alpha_2 + \cdots + p_{n2}\alpha_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_n = p_{1n}\alpha_1 + p_{2n}\alpha_2 + \cdots + p_{nn}\alpha_n \end{cases},$$

称此公式为基变换公式.

由于
$$\begin{cases} \beta_1 = p_{11}\alpha_1 + p_{21}\alpha_2 + \cdots + p_{n1}\alpha_n \\ \beta_2 = p_{12}\alpha_1 + p_{22}\alpha_2 + \cdots + p_{n2}\alpha_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_n = p_{1n}\alpha_1 + p_{2n}\alpha_2 + \cdots + p_{nn}\alpha_n \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \Leftrightarrow P^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

$$\Rightarrow \boxed{(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)P}$$

基变换公式

在基变换公式

$$(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)P$$

中, 矩阵 P 称为由基 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 到基 $\beta_1,\beta_2,\cdots,\beta_n$ 的过渡矩阵.

过渡矩阵 P 是可逆的.

二、坐标变换公式

定理1 设V_n中的元素α,在基$\alpha_1,\alpha_2,\cdots,\alpha_n$下的坐标为

$$(x_1,x_2,\cdots,x_n)^T,$$

在基$\beta_1,\beta_2,\cdots,\beta_n$下的坐标为

$$(x_1',x_2',\cdots,x_n')^T,$$

若两个基满足关系式

$$(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)P$$

则有坐标变换公式

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}, \quad \text{或} \quad \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = P^{-1}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

证明

$$\because \alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=(\beta_1,\beta_2,\cdots,\beta_n)\begin{pmatrix}x_1'\\x_2'\\\vdots\\x_n'\end{pmatrix}$$

$$(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)P$$

$$\therefore(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=(\alpha_1,\alpha_2,\cdots,\alpha_n)P\begin{pmatrix}x_1'\\x_2'\\\vdots\\x_n'\end{pmatrix}.$$

即
$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}.$$

由于矩阵P可逆,所以
$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

例1 在$P[x]_3$中取两个基

$$\alpha_1 = x^3 + 2x^2 - x, \qquad \alpha_2 = x^3 - x^2 + x + 1,$$

$$\alpha_3 = -x^3 + 2x^2 + x + 1, \quad \alpha_4 = -x^3 - x^2 + 1,$$

及 $$\beta_1 = 2x^3 + x^2 + 1, \qquad \beta_2 = x^2 + 2x + 2,$$

$$\beta_3 = -2x^3 + x^2 + x + 2, \quad \beta_4 = x^3 + 3x^2 + x + 2,$$

求坐标变换公式.

解 将$\beta_1,\beta_2,\beta_3,\beta_4$用$\alpha_1,\alpha_2,\alpha_3,\alpha_4$表示.

因为 $$(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (x^3, x^2, x, 1)A,$$

$$(\beta_1,\beta_2,\beta_3,\beta_4) = (x^3, x^2, x, 1)B,$$

其中 $A=\begin{pmatrix}1 & 1 & -1 & -1\\ 2 & -1 & 2 & -1\\ -1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\end{pmatrix}, B=\begin{pmatrix}2 & 0 & -2 & 1\\ 1 & 1 & 1 & 3\\ 0 & 2 & 1 & 1\\ 1 & 2 & 2 & 2\end{pmatrix},$

得 $$(\beta_1,\beta_2,\beta_3,\beta_4)=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)A^{-1}B.$$

故坐标变换公式为

$$\begin{pmatrix}x_1'\\ x_2'\\ x_3'\\ x_4'\end{pmatrix}=B^{-1}A\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}.$$

用初等变换计算 $B^{-1}A$.

$$(B \mid A)=\left(\begin{array}{cccc|cccc}
2 & 0 & -2 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 3 & 2 & -1 & 2 & -1 \\
0 & 2 & 1 & 1 & -1 & 1 & 1 & 0 \\
1 & 2 & 2 & 2 & 0 & 1 & 1 & 1
\end{array}\right)$$

$$\overset{\text{初等行变换}}{\sim}\left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & -1 & 1 & -1
\end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1 \end{array}\right) = \left(E \mid B^{-1}A\right)$$

所以
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

例2　坐标变换的几何意义.

设 $\alpha_1=\begin{pmatrix}1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\end{pmatrix}$

及 $\beta_1=\begin{pmatrix}1\\1\end{pmatrix}$, $\beta_2=\begin{pmatrix}1\\-1/2\end{pmatrix}$

为线性空间$V=R^2$的两个基.

又设$\alpha=-\dfrac{1}{2}\alpha_1+\alpha_2$, 则$\alpha$在基$\alpha_1,\alpha_2$下的坐标为

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-1/2\\1\end{pmatrix}$$

由坐标变换公式可知,α在基 β_1,β_2 下的坐标为

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1/2 \end{pmatrix}^{-1} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$$

即 $\alpha = \frac{1}{2}\beta_1 - \beta_2.$