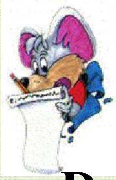
Section 2 Eigenvalues and Eigenvectors of Square Matrices



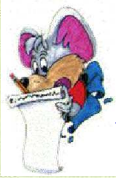
1. The concept of eigenvalues and eigenvectors

Definition 1. Let A be a square matrix of order n. If there exist a number λ and a non - zero vector x such that $Ax = \lambda x$, then the number λ is called an eigenvalue of A, and the vector x an eigenvector of A corresponding to the eigenvalue λ.

For example, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \end{pmatrix}$; hence,

2 is an eigenvalue of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ corresponding to 2.



2. How to Find Eigenvalues and Eigenvectors

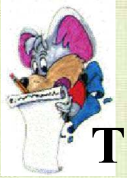
Since $Ax = \lambda x (x \neq 0)$, $(A - \lambda E)x = 0.$

Hence the system of homogeneou s linear equations $(A - \lambda E)x = 0$ **in** λ **has a non - zero solution .**

In other words, values of λ **should satisfy** $|A - \lambda E| = 0$ **.**

$$|A - \lambda E| = 0$$

$$\Leftrightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

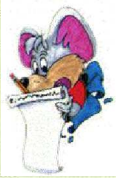


This is an n - th degree equation in λ, which will be called the characteristic equation of A.

Write $f(\lambda)=|A-\lambda E|$, which is a polynomial of degree n. $f(\lambda)$ will be referred to as the characteristic polynomial of A. It follows that there are n eigenvalues in the field of real numbers (repeated eigenvalue is counted as the multiplicty of the root).

Assume $\lambda_1,\lambda_2,\cdots,\lambda_n$ are all the chracteristic roots. Then

$$|\lambda E-A|=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$



On the other hand, by properties of determinants,

$$|\lambda E - A| = \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots + (-1)^n |A|$$

Therefore, the following conclusions can be drawn:

(1) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$;

(2) $\lambda_1 \lambda_2 \cdots \lambda_n = |A|$.

It follows that $|A| = 0$ iff $\lambda = 0$ is an eigenvalue iff $Ax = 0$ has a non-zero solution.

Example 1.

Find eigenvalues and eigenvectors of $A=\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}$.

Solution. **The characteristic polynomial of** A **is**

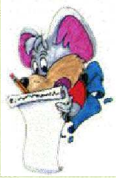
$$\begin{vmatrix}3-\lambda & -1\\ -1 & 3-\lambda\end{vmatrix}=(3-\lambda)^2-1$$

$$=8-6\lambda+\lambda^2=(4-\lambda)(2-\lambda)$$

Hence the eigenvalues of A **are** $\lambda_1=2, \lambda_2=4.$

For $\lambda_1=2,$ **the corresponding eigenvectors should**

satisfy that $\begin{pmatrix}3-2 & -1\\ -1 & 3-2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$



or equivalently $\begin{cases} x_1 - x_2 = 0, \\ -x_1 + x_2 = 0. \end{cases}$

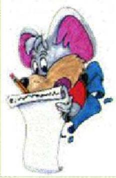
Solving the equations, we have $x_1 = x_2$.

For instance, $p_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

For $\lambda_2 = 4$,

$$\begin{pmatrix} 3-4 & -1 \\ -1 & 3-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

We obtain $x_1 = -x_2$, and thus $p_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector.



Example 2.

Find eigenvalues and eigenvectors of $A = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Solution.

The characteristic polynomial of A **is**

$$|A - \lambda E| = \begin{vmatrix} -1-\lambda & 1 & 0 \\ -4 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)^2,$$

and thus the eigenvalues of A **are** $\lambda_1 = 2, \lambda_2 = \lambda_3 = 1.$

For $\lambda_1 = 2,$ **solve the linear system** $(A - 2E)x = 0.$

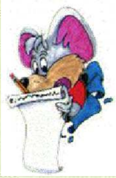
$$A-2E=\begin{pmatrix}-3 & 1 & 0\\ -4 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},$$

A system of fundamental solutions is $p_1=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

Hence $k\,p_1(k\neq 0)$ **are all the eigenvectors corresponding to the eigenvalue** $\lambda_1=2$.

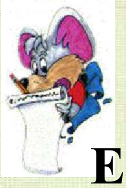
For $\lambda_2=\lambda_3=1$, **solve the system** $(A-E)x=0$.

$$A-E=\begin{pmatrix}-2 & 1 & 0\\ -4 & 2 & 0\\ 1 & 0 & 1\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix},$$



A system of fundamental solutions is $p_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

$k\,p_2 (k \neq 0)$ are all the eigenvectors corresponding to the eigenvalue $\lambda_2 = \lambda_3 = 1$.



Example 3. Let $A = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 2 & 0 \\ -4 & 1 & 3 \end{pmatrix}$.

Find the eigenvalues and eigenvectors of A.

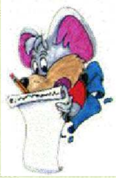
Solution.

$$|A - \lambda E| = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ -4 & 1 & 3-\lambda \end{vmatrix}$$

$$= -(\lambda + 1)(\lambda - 2)^2.$$

From $-(\lambda + 1)(\lambda - 2)^2 = 0,$

it follows that $\lambda_1 = -1, \lambda_2 = \lambda_3 = 2.$



For $\lambda_1 = -1$, solve $(A+E)x = 0$.

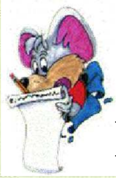
$$A + E = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 0 \\ -4 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

We have a system of fundamental solutions

$$p_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvecto rs correspond ing to $\lambda_1 = -1$ are

$$k\ p_1 \qquad (k \neq 0).$$



For $\lambda_2 = \lambda_3 = 2$, solve $(A - 2E)x = 0$.

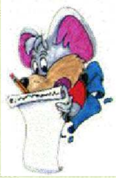
$$A - 2E = \begin{pmatrix} -4 & 1 & 1 \\ 0 & 0 & 0 \\ -4 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A system of fundamental solutions is

$$p_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix},$$

The eigenvectors corresponding to $\lambda_2 = \lambda_3 = 2$ are

$$k_2 p_2 + k_3 p_3 \quad (k_2 k_3 \neq 0).$$



2.Properties of eigenvalues and eigenvectors

Theorem 1. If $\lambda_1, \lambda_2, \cdots, \lambda_m$ are m distinct eigenvalues of A with corresponding eigenvectors $p_1, p_2, \cdots, p_m$ res., then $p_1, p_2, \cdots, p_m$ are linearly independent.

Proof. Let $x_1, x_2, \cdots, x_m$ be such that

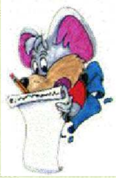
$$x_1 p_1 + x_2 p_2 + \cdots + x_m p_m = 0.$$

Then $A(x_1 p_1 + x_2 p_2 + \cdots + x_m p_m) = 0,$ and thus

$$\lambda_1 x_1 p_1 + \lambda_2 x_2 p_2 + \cdots + \lambda_m x_m p_m = 0.$$

Following the same reasoning, we have

$$\lambda_1^k x_1 p_1 + \lambda_2^k x_2 p_2 + \cdots + \lambda_m^k x_m p_m = 0. \ (k = 1,2,\cdots,m-1)$$



Merge them to write in the form of matrix.

$$\left(x_1p_1, x_2p_2, \cdots, x_mp_m\right)\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \cdots & \lambda_m^{m-1} \end{pmatrix} = \left(0,0,\cdots,0\right)$$

The determinant of the second matrix on the left - hand side is the Vandemonde determinant whose value is non - zero for $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct.

Hence the matrix is invertible. As a result,

$$\left(x_1p_1, x_2p_2, \cdots, x_mp_m\right) = \left(0,0,\cdots,0\right),$$

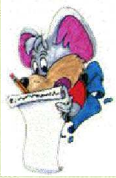
i.e. $x_jp_j = 0\left(j = 1,2,\cdots,m\right).$

From $p_j \neq 0$, it follows that $x_j = 0\,(j = 1, 2, \cdots, m)$.

Thus $p_1, p_2, \cdots, p_m$ are linearly independent.

Remarks.

1. **The eigenvectors corresponding to distinct eigenvalues are linearly independent.**
2. **Every non-zero linear combination of eigenvectors corresponding to the same eigenvalue is still an eigenvector corresponding to the eigenvalue.**
3. **An eigenvector corresponds to a unique eigenvalue.**



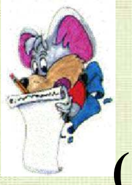
Otherwise, suppose that x is an eigenvector of A corresponding to two distinct eigenvalues λ_1, λ_2 then we have successively the following

$$Ax = \lambda_1 x, \qquad Ax = \lambda_2 x$$

$$\Rightarrow \lambda_1 x = \lambda_2 x$$

$$\Rightarrow (\lambda_1 - \lambda_2) x = 0,$$

Hence $x = 0$ due to $\lambda_1 \neq \lambda_2$, which contradicts the definition of eigenvector.



Theorem 2. Let λ be an eigenvalue of A. Then

(1) A^T and A have the same chracteristic polynomial, and thus the same chracteristic values.

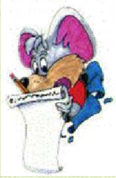
(2) $k\lambda$ is an eigenvalue of kA.

(3) λ^m is an eigenvalue of A^m (m is an arbitrary positive integer).

(4) $\varphi(\lambda)$ is an eigenvalue of $\varphi(A)$, where $\varphi(\lambda)$ is a polynomial in λ.

(5) If A is invertible, then λ^{-1} is an eigenvalue of A^{-1}.

(6) If $\lambda \neq 0$, then $\frac{|A|}{\lambda}$ is an eigenvalue of A^*.



Proof.

(1) $f_{A^T}(\lambda) = \left|\lambda E - A^T\right| = \left|(\lambda E - A)^T\right| = \left|\lambda E - A\right| = f_A(\lambda).$

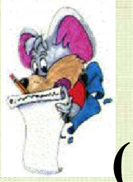
(2) Let $Ax = \lambda x$. Then $(kA)x = k(Ax) = (k\lambda)x$,

$i.e.$ $k\lambda$ is an eigenvalue of kA.

(3) From $Ax = \lambda x$ it follows that

$$A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x.$$

Repeat this procedure by another m-2 times, we have $A^m x = \lambda^m x$, i.e. λ^m is an eigenvalue of A^m.



(4) Let $\varphi(y) = b_k y^k + b_{k-1} y^{k-1} + \cdots + b_1 y + b_0$. Then

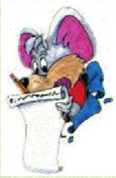
$$\begin{aligned}\varphi(A)x &= (b_k A^k + b_{k-1} A^{k-1} + \cdots + b_1 A + b_0 E)x \\ &= b_k A^k x + b_{k-1} A^{k-1} x + \cdots + b_1 Ax + b_0 x \\ &= b_k \lambda^k x + b_{k-1} \lambda^{k-1} x + \cdots + b_1 \lambda x + b_0 x \\ &= (b_k \lambda^k + b_{k-1} \lambda^{k-1} + \cdots + b_1 \lambda + b_0) x = \varphi(\lambda) x\end{aligned}$$

(5) If A is invertible, then it is clear that $\lambda \neq 0$.

From $Ax = \lambda x$, it follows that

$$A^{-1}(Ax) = A^{-1}(\lambda x) = \lambda A^{-1} x \text{ and thus } A^{-1} x = \lambda^{-1} x.$$

In other words, λ^{-1} is an eigenvalue of A^{-1}.



(6) Since $Ax = \lambda x$ and $A^* A = |A| E$,

$$A^* A x = A^* \lambda x = \lambda A^* x = |A| x$$

Therefore, $A^* x = \dfrac{|A|}{\lambda} x.$