Section 1 Inner Product of Vectors

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1. Definition and properties

Definition 1. Given two n-dimensional vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

then $[x, y] = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

is called the inner product of x and y.

Remarks.

(1) **Although the inner product of two n-dimensional vectors for $n \geq 4$ is a generlization of the same notion in the 3-dimensional space, it is no longer intepreted in an intuitive way as in the 3-dimensional space.**

(2) **Inner product is a kind of operations on vectors. If both x and y are column vectors, their inner product can be denoted by $[x, y] = x^T y$.**

Properties of inner product.

Let x, y, z be vectors of the same size and λ a number.

(1) $[x, y] = [y, x]$;

(2) $[\lambda x, y] = \lambda [x, y]$;

(3) $[x + y, z] = [x, z] + [y, z]$;

(4) $[x, x] \geq 0$, and $[x, x] > 0$ iff $x \neq 0$.

2. The magnitude of a vector and its properties

Definition 2. Let $x=(x_1,x_2,\cdots,x_n)$.

Then $\|x\|=\sqrt{[x,x]}=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$,

is called the magnitude of x, or the norm of x.

The magnitude is of the following properties:

1. Non - negativity :

   If $x\neq 0$, then $\|x\|>0$; if $x=0$, then $\|x\|=0$.

2. Homogeneity : $\|\lambda x\|=|\lambda|\|x\|$;

3. Triangle inequality : $\|x+y\|\leq\|x\|+\|y\|$.

Unit vectors and angles between vectors.

(1) If $\|x\| = 1$, then x is called a unit vector.

(2) If $\|x\| \neq 0, \|y\| \neq 0$, then $\theta = \arccos\dfrac{[x, y]}{\|x\|\|y\|}$

is called the angle beteen x and y.

Example. Find the angle beteen $\alpha = (1,2,2,3)$ and $\beta = (3,1,5,1)$.

Solution. Since $\cos\theta = \dfrac{\alpha \cdot \beta}{\|\alpha\|\|\beta\|} = \dfrac{18}{3\sqrt{2}\cdot 6} = \dfrac{\sqrt{2}}{2}$,

$$\theta = \frac{\pi}{4}.$$

3.Normal Orthogonal bases

(1) Orthogonality

If $[x, y] = 0$, then x and y is called orthogonal. .

By definition, the vector 0 is orthogonal to any other vector.

(2) Orthogonal set of vectors

If each pair of vectors in a set of vectors is orthogonal, then the set of vectors is called orthogonal.

3. A property of orthogonal set of vectors

Theorem 1. If the non - zero vectors $\alpha_1, \alpha_2, \cdots, \alpha_r$ are mutually orthogonal then $\alpha_1, \alpha_2, \cdots, \alpha_r$ are linearly independent.

Proof. Assume that there are $\lambda_1, \lambda_2, \cdots, \lambda_r$ such that

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_r\alpha_r = 0.$$

Multiplyin g it by a_1^T leads to $\lambda_1 {\alpha_1}^T \alpha_1 = 0.$

It follows from $\alpha_1 \neq 0$ that ${\alpha_1}^T \alpha_1 = \|\alpha_1\|^2 \neq 0,$

and thus $\lambda_1 = 0.$ Similarly, $\lambda_2 = \cdots = \lambda_r = 0.$

Therefore, $\alpha_1, \alpha_2, \cdots, \alpha_r$ are linearly independent.

4. Orthogonal bases of a vector space

If $\alpha_1, \alpha_2, \cdots, \alpha_r$ is a basis of the vector space V and if $\alpha_1, \alpha_2, \cdots, \alpha_r$ are non - zero vectors, then $\alpha_1, \alpha_2, \cdots, \alpha_r$ is called an orthogonal basis of V.

Example 1. Given two orthogonal vectors of the 3-dimensional space

$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Find a vector α_3 such that $\alpha_1, \alpha_2, \alpha_3$ form an orthogonal base of the 3-dimensional space.

Solution.

Assume that $\alpha_3 = (x_1, x_2, x_3)^T \neq 0$ **is orthogonal to** α_1, α_2.

Then $[\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = 0$, **namely**

$$\begin{cases} [\alpha_1, \alpha_3] = x_1 + x_2 + x_3 = 0 \\ [\alpha_2, \alpha_3] = x_1 - 2x_2 + x_3 = 0 \end{cases}$$

Therefore $x_1 = -x_3,\ x_2 = 0.$

By putting $x_3 = 1$, **we have** $\alpha_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

which can serve as the vector desired.

(5) Normal orthogonal bases

Definition 3. Let $e_1, e_2, \cdots, e_r$ be a base of the vector space $V \subset R^n$). If all of them are unit vectors and if they are mutually orthogonal, then $e_1, e_2, \cdots, e_r$ is called a normal orthogonal (or shortly, orthonormal) basis of V.

For example,

$$e_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, e_4 = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

$$e_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, e_4 = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

Since $\begin{cases} [e_i, e_j] = 0, & i \neq j;\ i, j = 1,2,3,4. \\ [e_i, e_j] = 1, & i = j;\ i, j = 1,2,3,4. \end{cases}$

e_1, e_2, e_3, e_4 **is an orthonorma l basis of** R^4.

Following the same reasoning,

$$\varepsilon_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\varepsilon_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\varepsilon_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\varepsilon_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

is also an orthonormal basis of R^4**.**

6. An approach to finding an orthonormal basis

Let $\alpha_1,\alpha_2,\cdots,\alpha_r$ be a basis of V. Our aim is to find an orthonormal basis of V such that $e_1,e_2,\cdots,e_r$ and $\alpha_1,\alpha_2,\cdots,\alpha_r$ are equivalent. This process is said to orthonormalize $\alpha_1,\alpha_2,\cdots,\alpha_r$.

Assume that $a_1,a_2,\cdots,a_r$ is a basis of V.

（1） Orthogonalization.

Let $b_1 = a_1,\quad b_2 = a_2 - \dfrac{[b_1,a_2]}{[b_1,b_1]}b_1,$

$$b_3 = a_3 - \frac{[b_1, a_3]}{[b_1, b_1]} b_1 - \frac{[b_2, a_3]}{[b_2, b_2]} b_2$$

$$\cdots\cdots\cdots\cdots$$

$$b_r = a_r - \frac{[b_1, a_r]}{[b_1, b_1]} b_1 - \frac{[b_2, a_r]}{[b_2, b_2]} b_2 - \cdots - \frac{[b_{r-1}, a_r]}{[b_{r-1}, b_{r-1}]} b_{r-1}$$

Thus $b_1, \cdots, b_r$ are mutually orthogonal, and $b_1, \cdots, b_r$ and $a_1, \cdots a_r$ are equivalent.

（2）Normalization. Divide each vector by its norm.

$$e_1 = \frac{b_1}{\|b_1\|}, \quad e_2 = \frac{b_2}{\|b_2\|}, \quad \cdots\cdots, e_r = \frac{b_r}{\|b_r\|}.$$

Clearly, $e_1, e_2, \cdots, e_r$ is an orthogonal basis of V.

The above method to contruct an orthogonal set of vectors $b_1,\cdots,b_r$ by using linearly independent vectors $a_1,\cdots,a_r$ is called the Gram - Schmidt orthogonalization process.

Example 2. Apply the Gram-Schmidt orthogonalization process to orthonormalize the following set of vectors.

$$a_1 = (1,1,1,1), a_2 = (1,-1,0,4), a_3 = (3,5,1,-1)$$

Solution. Firstly, we orthogonalize them.

Let $b_1 = a_1 = (1,1,1,1)$. **Then**

$$b_2 = a_2 - \frac{[b_1, a_2]}{[b_1, b_1]} b_1$$

$$= (1,-1,0,4) - \frac{1-1+4}{1+1+1+1}(1,1,1,1) = (0,-2,-1,3)$$

$$b_3 = a_3 - \frac{[b_1, a_3]}{[b_1, b_1]} b_1 - \frac{[b_2, a_3]}{[b_2, b_2]} b_2$$

$$= (3,5,1,-1) - \frac{8}{4}(1,1,1,1) - \frac{-14}{14}(0,-2,-1,3) = (1,1,-2,0)$$

Then we normalize b_1, b_2, b_3.

$$e_1 = \frac{b_1}{\|b_1\|} = \frac{1}{2}(1,1,1,1) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$e_2 = \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{14}}(0,-2,-1,3) = \left(0, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

$$e_3 = \frac{b_3}{\|b_3\|} = \frac{1}{\sqrt{6}}(1,1,-2,0) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, 0\right)$$

Example 3.

Let $a_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, a_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$**. Apply the Gram - schmidt orthogonalization process to get orthonormal vectors.**

Solution. **By letting** $b_1 = a_1$**, we have**

$$b_2 = a_2 - \frac{[a_2, b_1]}{\|b_1\|^2} b_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \frac{4}{6}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{5}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix};$$

$$b_3 = a_3 - \frac{[a_3, b_1]}{\|b_1\|^2} b_1 - \frac{[a_3, b_2]}{\|b_2\|^2} b_2$$

$$= \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Dividing each vector by its norm, we have

$$e_1 = \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad e_2 = \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

$$e_3 = \frac{b_3}{\|b_3\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

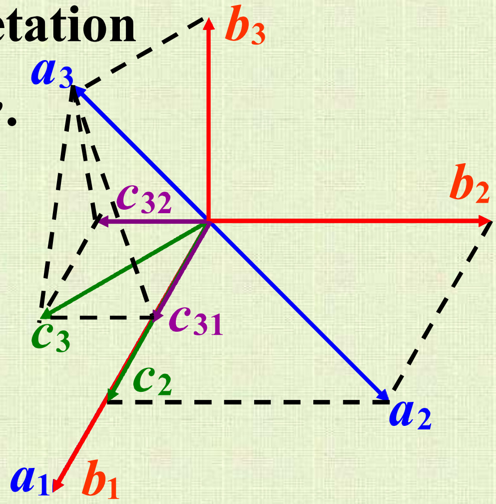
e_1, e_2, e_3 **is the vectors desired.**

$b_1 = a_1$; Geometric interpretation

c_2 is the projection of a_2 on b_1, *i.e.*

$$c_2 = [a_2, \frac{b_1}{\|b_1\|}]\frac{b_1}{\|b_1\|} = \frac{[a_2, b_1]}{\|b_1\|^2} b_1,$$

$$b_2 = a_2 - c_2;$$



c_3 is the projection of a_3 on the plane spanned by b_1, b_2.

Since $b_1 \perp b_2$, c_3 is the sum of projections c_{31} and c_{32} of a_3 on b_1, b_2, i.e.

$$c_3 = c_{31} + c_{32} = \frac{[a_3, b_1]}{\|b_1\|^2} b_1 + \frac{[a_3, b_2]}{\|b_2\|^2} b_2, \qquad b_3 = a_3 - c_3.$$

Example 4.

Given $a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$**, find the non - zero vectors** a_2, a_3 **such that** a_1, a_2, a_3 **are mutually orthogonal.**

Solution. a_2, a_3 **should satisfy the equaion** $a_1^T x = 0$**, or**

$$x_1 + x_2 + x_3 = 0.$$

Its one of systems of fundamental solutions is

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Orthogonalize vectors in the system of fundamental solutions in order to get the desired vectors.

Let $a_2 = \xi_1,\ a_3 = \xi_2 - \dfrac{[\xi_1, \xi_2]}{[\xi_1, \xi_1]}\xi_1,$

where $[\xi_1, \xi_2] = 1, [\xi_1, \xi_1] = 2.$

$$a_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\quad a_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

4 Orthogonal matrices and orthogonal transformations

Definition 4. If a square matrix A satisfies that $A^T A = E \left(\text{or } A^{-1} = A^T\right)$, then A is said to be an orthogonal matrix .

Theorem. A is an orthogonal matrix iff all of its column vectors are unit vectors and they are mutually orthogonal.

Proof.

$$AA^T = E \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} = E$$

$$\Leftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \left(\alpha_1^T, \alpha_2^T, \cdots, \alpha_n^T\right) = E$$

$$\Leftrightarrow \begin{pmatrix} \alpha_1 \alpha_1^T & \alpha_1 \alpha_2^T & \cdots & \alpha_1 \alpha_n^T \\ \alpha_2 \alpha_1^T & \alpha_2 \alpha_2^T & \cdots & \alpha_2 \alpha_n^T \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_n \alpha_1^T & \alpha_n \alpha_2^T & \cdots & \alpha_n \alpha_n^T \end{pmatrix} = E$$

$$\Leftrightarrow \alpha_i \alpha_j^T = \delta_{ij} = \begin{cases} 1, \text{if } i = j; \\ 0, \text{otherwise} \end{cases} \quad (i, j = 1, 2, \cdots, n)$$

Remarks.

1. Since $A^{-1} = A^T$ for the orthogonal matrix A, $AA^T = E$. As a result, all row vectors of an orthogonal matrix are unit vectors and mutually orthogonal.

2. Since an orthogonal matrix satisfies $A^T A = E$, $|A^T||A| = 1$, i.e. $|A| = \pm 1$. Hence every orthogonal matrix is invertible.

Definition 5. If P is an orthogonal matrix, the linear transformation $y=Px$ is called an orthogonal transformation.

Property. An orthogonal transformation does not change the magnitude of a vector.

Proof. Let $y = Px$ be the orthogonal transformation.

Then $\|y\| = \sqrt{y^T y} = \sqrt{x^T P^T Px} = \sqrt{x^T x} = \|x\|.$

Example 5. Whether are the following matrices orthogonal or not?

$$(1)\begin{pmatrix} 1 & -1/2 & 1/3 \\ -1/2 & 1 & 1/2 \\ 1/3 & 1/2 & -1 \end{pmatrix}, \qquad (2)\begin{pmatrix} \frac{1}{9} & -\frac{8}{9} & -\frac{4}{9} \\ -\frac{8}{9} & \frac{1}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{4}{9} & \frac{7}{9} \end{pmatrix}.$$

Solution. $(1)\begin{pmatrix} 1 & -1/2 & 1/3 \\ -1/2 & 1 & 1/2 \\ 1/3 & 1/2 & -1 \end{pmatrix}$

Pay attention to the first and second column.

Since $1\times\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)\times 1+\frac{1}{3}\times\frac{1}{2}\neq 0,$

the matrix is not orthogonal.

(2) $\begin{pmatrix} \frac{1}{9} & -\frac{8}{9} & -\frac{4}{9} \\ -\frac{8}{9} & \frac{1}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{4}{9} & \frac{7}{9} \end{pmatrix}$

Since

$$\begin{pmatrix} \frac{1}{9} & -\frac{8}{9} & -\frac{4}{9} \\ -\frac{8}{9} & \frac{1}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{4}{9} & \frac{7}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{9} & -\frac{8}{9} & -\frac{4}{9} \\ -\frac{8}{9} & \frac{1}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{4}{9} & \frac{7}{9} \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the matrix is orthogonal.

Example 6.

Show that

$$P=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ is an orthogonal matrix.}$$

Proof.

Since all column vectors of P are unit vectors and the set of column vectors is orthogonal, P is orthogonal.