Section 4. Linear Transformation and Its Matrix Expression

4

1 The concept of linear transformation

1. Mapping

Definition 1. Let A, B be two non-empty sets. To each α in A, by some rule T, corresponds one and only one definite β in B. Then the rule T is called a mapping from A to B, which is denoted by $\beta = T(\alpha)$ or $\beta = T\alpha$, $(\alpha \in A)$. β is called the image of α and α is called the pre-image of β.

For $A_0 \subseteq A$, the set of the images of all elements in A_0 is denotedd by $T(A)$, i.e.

$$T(A_0) = \{\beta = T(\alpha) | \alpha \in A_0\},$$

Apparently, $T(A) \subset B$.

Clearly, mapping is an extension of the concept of function.

Definition 2. Let V be a linear space T a mapping from V to V. If T satisfies the following conditions, then T is called a linear transformation.

(1) $\forall \alpha, \beta \in V,\ T(\alpha+\beta)=T(\alpha)+T(\beta).$

(2) $\forall\ \lambda \in R, \alpha \in V, T(\lambda\alpha)=\lambda T(\alpha).$

Example 1. In $P[x]_3$, the differentiation operation is a linear transformation.

Proof. $\forall \quad p=a_3x^3+a_2x^2+a_1x+a_0 \in P[x]_3,$

$$Dp=3a_3x^2+2a_2x+a_1,$$

$$q = b_3 x^3 + b_2 x^2 + b_1 x + b_0 \in P[x]_3,$$

$$Dq = 3b_3 x^2 + 2b_2 x + b_1,$$

Hence, $D(p+q)$

$$= D[(a_3 + b_3)x^3 + (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)]$$

$$= 3(a_3 + b_3)x^2 + 2(a_2 + b_2)x + (a_1 + b_1)$$

$$= (3a_3 x^2 + 2a_2 x + a_1) + (3b_3 x^2 + 2b_2 x + b_1)$$

$$= Dp + Dq;$$

$$D(kp) = D(k a_3 x^3 + k a_2 x^2 + k a_1 x + k a_0)$$
$$= k(3 a_3 x^2 + 2 a_2 x + a_1)$$
$$= kDp.$$

Hence, T is a linear transformation.

Remark 1. **If $T(p) = a_o$, then T is also a linear transformation.**

The reason: $T(p+q) = a_0 + b_0 = T(P) + T(q).$

$$T(\lambda p) = \lambda a_0 = \lambda T(p).$$

Remark 2. If $T(p)=1$, then T is not a linear transformation.

The reason: $T(p+q)=1$,

but $T(p)+T(q)=1+1=2$,

As a result, $T(p+q)\neq T(p)+T(q)$.

Example 2. Let T be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

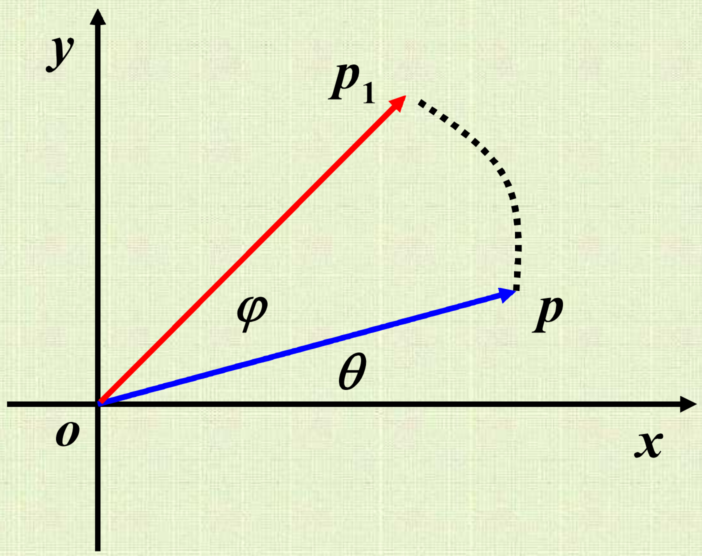
Then T is a linear transformation over xOy plane.

Solution. Let $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases}$ so

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\varphi - y\sin\varphi \\ x\sin\varphi + y\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} r\cos\theta\cos\varphi - r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi + r\sin\theta\cos\varphi \end{pmatrix} = \begin{pmatrix} r\cos(\theta+\varphi) \\ r\sin(\theta+\varphi) \end{pmatrix},$$

Geometric explanation : T **rotates a vector by an angle of** φ **anti-clockwise.**



Example 3. The set of all continuous functions on the interval [*a,b*] is a linear space. In this space, define

$$T(f(x)) = \int_a^x f(t)dt$$

Then *T* is a linear transformation.

Proof. Let $f(x) \in V,\ g(x) \in V$.

Then $$T[f(x)+g(x)] = \int_a^x f(t)+g(t)dt$$

$$= \int_a^x f(t)dt + \int_a^x g(t)dt$$

$$= T[f(x)] + T[g(x)]$$

$$T(kf(x)) = \int_a^x kf(t)dt = k\int_a^x f(t)dt = kT[f(x)].$$

Example 4. **The following mapping in any linear space V is a linear transformation.**

$$E(\alpha) = \alpha, \quad \alpha \in V.$$

Proof. **For any $\alpha, \beta \in V$,**

$$E(\alpha + \beta) = \alpha + \beta = E(\alpha) + E(\beta)$$

$$E(k\alpha) = k\alpha = kE(\alpha).$$

Hence E is a linear transformation, which is called identity transformation.

Example 5. **The zero transformation in any linear space V defined by $O(\alpha) = 0 (\forall \alpha \in V)$ is a linear transformation.**

Proof. **Let $\alpha, \beta \in V$. Then**

$$0(\alpha + \beta) = 0 = 0 + 0 = 0(\alpha) + 0(\beta)$$

$$0(k\alpha) = 0 = k0 = k0(\alpha).$$

So the zero transformation is linear.

Example 6. In R^3, define

$$T(x_1, x_2, x_3) = (x_1^2, x_2 + x_3, 0)$$

Then T is not a linear transformation.

Proof. $\forall \alpha = (a_1, a_2, a_3), \beta = (b_1, b_2, b_3) \in R^3$,

$$\begin{aligned} T(\alpha + \beta) &= T(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= ((a_1 + b_1)^2, a_2 + a_3 + b_2 + b_3, 0) \\ &\neq (a_1^2, a_2 + a_3, 0) + (b_1^2, b_2 + b_3, 0) \\ &= T(\alpha) + T(\beta). \end{aligned}$$

2. Properties of linear transformations

1. $T(0) = 0, \quad T(-\alpha) = -T(\alpha);$

2. If $\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m$, then

$$T\beta = k_1 T\alpha_1 + k_2 T\alpha_2 + \cdots + k_m T\alpha_m;$$

3. If $\alpha_1, \alpha_2, \cdots, \alpha_m$ is linearly dependent, then $T\alpha_1, T\alpha_2, \cdots, T\alpha_m$ is also linearly dependent .

Note. Linear independen ce of $\alpha_1, \alpha_2, \cdots, \alpha_m$ does not necessaril y imply linear independen ce of $T\alpha_1, T\alpha_2, \cdots, T\alpha_m$.

4. Let T be a linear transformation of V_n. Then the set $T(V_n)$ of the images of V_n is a subspace of V_n.

Proof. Let $\beta_1, \beta_2 \in T(V_n)$, so there exist $\alpha_1, \alpha_2 \in V_n$ such that $T\alpha_1 = \beta_1, T\alpha_2 = \beta_2$. Thus

$$\beta_1 + \beta_2 = T\alpha_1 + T\alpha_2 = T(\alpha_1 + \alpha_2) \in T(V_n).$$

$$k\beta_1 = kT\alpha_1 = T(k\alpha_1) \in T(V_n).$$

Since $T(V_n) \subset V_n$, it is a subspace of V_n.

5. Let $S_T = \{\alpha | \alpha \in V_n, T\alpha = 0\}$. Then S_T is a subspace of V_n which is called the kernel of T.

Proof. If $\alpha_1, \alpha_2 \in S_T$, then $T\alpha_1 = 0, T\alpha_2 = 0$,

and thus $T(\alpha_1 + \alpha_2) = T\alpha_1 + T\alpha_2 = 0$ i.e. $\alpha_1 + \alpha_2 \in S_T$.

If $\alpha_1 \in S_T, k \in R$, then

$$T(k\alpha_1) = kT\alpha_1 = k0 = 0, \text{ i.e. } k\alpha_1 \in S_T.$$

As a result, S_T is closed for the linear operations.

From $S_T \subset V_n$, it follows that S_T is a subspace of V_n.

Example 7. Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}=(\alpha_1,\alpha_2,\cdots,\alpha_n),$$

where $\alpha_i=\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix}$. Define a mapping T in R^n by

$$T(x)=Ax(x\in R^n).$$

Then T is a linear transformation .

Indeed, if $a, b \in R^n$, then

$$T(a+b) = A(a+b) = Aa + Ab = T(a) + T(b);$$

$$T(ka) = A(ka) = kAa = kT(a).$$

Clearly,

$$T(R^n) = \{y = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n \mid x_1, x_2, \cdots, x_n \in R\}$$

whereas the kernel S_T of T is the solution space of the system of linear equations $Ax = 0$.

3.The matrix expression of a linear transformation

As we know, if

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}=(\alpha_1,\alpha_2,\cdots,\alpha_n),$$

where $\alpha_i=\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix}$, define T by

$T(x) = Ax (x \in R^n)$, then T is a linear transformation.

Assume $e_1, e_2, \cdots, e_n$ are the unit coordinate vectors.

Then $$Ae_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \alpha_1, \quad \cdots\cdots,$$

$$Ae_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \alpha_n,$$

i.e. $\alpha_{\rm i} = Ae_i = T(e_i) \qquad (i = 1, 2, \cdots, n)$

Conclusion : If $T(x) = Ax$, **then the columns of** A **are** $T(e_i)(i = 1, 2, \cdots n)$.

Conversely, if a linear transformation T **such that** $T(e_i) = \alpha_i (i = 1, 2, \cdots, n)$, **then**

$$
\begin{aligned}
T(x) &= T[(e_1, e_2, \cdots, e_n)x] \\
&= T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\
&= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \\
&= (T(e_1), T(e_2), \cdots, T(e_n))x \\
&= (\alpha_1, \alpha_2, \cdots, \alpha_n)x = Ax.
\end{aligned}
$$

In other words,

Each linear transformation T of R^n can be expressed by

$$T(x) = Ax \qquad (x \in R^n)$$

where $A = (T(e_1), T(e_2), \cdots, T(e_n))$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

with $e_1, e_2, \cdots, e_n$ being the unit coordinate vectors.

4.The matrix of a linear transformation under a given basis

Definition 1. Let V be an n-dimensional linear space T a linear transformation of V. Given a basis of V, $\alpha_1, \alpha_2, \cdots, \alpha_n$, if

$$\begin{cases} T(\alpha_1) = a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{n1}\alpha_n, \\ T(\alpha_2) = a_{12}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{n2}\alpha_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T(\alpha_n) = a_{1n}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{nn}\alpha_n, \end{cases}$$

denoted by $T(\alpha_1,\alpha_2,\cdots,\alpha_n)=(T(\alpha_1),T(\alpha_2),\cdots,T(\alpha_n))$,

then we have

$$T(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A$$

where $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$,

We shall call A **the matrix of the linear transformation under the given basis** $\alpha_1,\alpha_2,\cdots,\alpha_n$

Obviously, A is uniquely determined by $T(\alpha_1),\cdots,T(\alpha_n)$.

Now assume A is the matrix of T under the given basis $\alpha_1,\alpha_2,\cdots,\alpha_n$, i.e.

$$T(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A.$$

Let us check what conditions should be satisfied by T.

$$\forall \alpha \in V_n, \text{let } \alpha = \sum_{i=1}^{n} x_i \alpha_i, \text{then}$$

$$T(\alpha) = T(\sum_{i=1}^{n} x_i \alpha_i) = \sum_{i=1}^{n} x_i T(\alpha_i)$$

$$= (T(\alpha_1), T(\alpha_2), \cdots, T(\alpha_n)) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= (\alpha_1, \alpha_2, \cdots, \alpha_n) A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

namely,

$$T\left[(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\right]=(\alpha_1,\alpha_2,\cdots,\alpha_n)A\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}.$$

The above expression uniquely detertermines the linear transformation T, whose matrix is A.

Conclusion:

When a basis in a linear space is selected, a linear transformation T uniquely determines a matrix A. Conversely, a matrix A uniquely determines a linear transformation T.

In other words, linear transformation and matrix are uniquely determined by each other after a basis is selected.

From

$$T\left[(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\right]=(\alpha_1,\alpha_2,\cdots,\alpha_n)A\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$

It follows that under $\alpha_1,\alpha_2,\cdots,\alpha_n$,

the coordinate of α **is** $\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$.

The coordinate of $T(\alpha)$ is $A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$,

or equivalently,

$$T(\alpha) = A\alpha.$$

Example 1. In $P[x]_4$, take the basis

$$p_1 = x^3, p_2 = x^2, p_3 = x, p_4 = 1.$$

Find the matrix of the differentiation operation D.

Solution.

$$\begin{cases} D\,p_1 = 3x^2 = 0\,p_1 + 3\,p_2 + 0\,p_3 + 0\,p_4, \\ D\,p_2 = 2x = 0\,p_1 + 0\,p_2 + 2\,p_3 + 0\,p_4, \\ D\,p_3 = 1 \quad = 0\,p_1 + 0\,p_2 + 0\,p_3 + 1\,p_4, \\ D\,p_4 = 0 \quad = 0\,p_1 + 0\,p_2 + 0\,p_3 + 0\,p_4, \end{cases}$$

Hence the matrix of *D* under the basis in question is

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Example 2.

In the linear space $R[x]_n$, define the mapping

$$\sigma(f(x)) = \frac{d}{dx} f(x), f(x) \in R[x]_n.$$

By properties of derivatives, σ is a linear transformation of $R[x]_n$, which is called differential transformation.

Take the basis $1, x, x^2, \cdots, x^{n-1}$ in $R[x]_n$.

Then we have

$$\sigma(1)=0,\qquad \sigma(x)=1,\qquad \sigma(x^2)=2x,$$

$$\cdots\cdots\cdots,\qquad \sigma(x^{n-1})=(n-1)\,x^{n-2}$$

So the matrix of σ under $1, x, x^2, \cdots, x^{n-1}$ is

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Example 3. In R^3, let T denote the projection on the xOy plane, i.e.

$$T(x\vec{i} + y\vec{j} + z\vec{k}) = x\vec{i} + y\vec{j},$$

(1)Find the matrix of T under the basis $\vec{i}, \vec{j}, \vec{k}$.

(2)Find the matrix of T under the basis $\alpha = \vec{i}, \beta = \vec{j}$, $\gamma = \vec{i} + \vec{j} + \vec{k}$.

Solution. (1)

$$\begin{cases} T\vec{i} = \vec{i}, \\ T\vec{j} = \vec{j}, \\ T\vec{k} = \vec{0}, \end{cases}$$

i.e. $$T(\vec{i}, \vec{j}, \vec{k}) = (\vec{i}, \vec{j}, \vec{k}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2) $$\begin{cases} T\alpha = \vec{i} = \alpha, \\ T\beta = \vec{j} = \beta, \\ T\gamma = \vec{i} + \vec{j} = \alpha + \beta, \end{cases}$$

i.e. $$T(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This example shows that a linear transformation has different matrix under different bases.

However, there exists some relationship among these matrices, which is explicitly shown in the following theorem.

5. The matrices of a linear transformation under different bases

Theorem 1. Let $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ and $\{\beta_1,\beta_2,\cdots,\beta_n\}$ be two bases in the linear space V_n. If the transition matrix from $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ to $\{\beta_1,\beta_2,\cdots,\beta_n\}$ is P, and if the matrices of a linear transformation T under $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ and $\{\beta_1,\beta_2,\cdots,\beta_n\}$ are A and B respectively, then we have

$$B = P^{-1}AP.$$

Proof. Since $(\beta_1, \beta_2, \cdots, \beta_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)P$

$$T(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)A,$$

$$T(\beta_1, \beta_2, \cdots, \beta_n) = (\beta_1, \beta_2, \cdots, \beta_n)B$$

Thus $(\beta_1, \beta_2, \cdots, \beta_n)B = T(\beta_1, \beta_2, \cdots, \beta_n)$

$$= T[(\alpha_1, \alpha_2, \cdots, \alpha_n)P]$$

$$= T[(\alpha_1, \alpha_2, \cdots, \alpha_n)]P$$

$$= (\alpha_1, \alpha_2, \cdots, \alpha_n) AP$$

$$= (\beta_1, \beta_2, \cdots, \beta_n) P^{-1} AP$$

Since $\beta_1, \beta_2, \cdots, \beta_n$ **are linearly independent,**

$$B = P^{-1} AP.$$

It follows from Theorem 1 that the matrices of a linear transformation under two bases are similar.

Example 4.

Let the matrix of a linear transformation T of V_2 under $\{\alpha_1, \alpha_2\}$ is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Find the matrix of T under the basis $\{\alpha_2, \alpha_1\}$.

Solution. $$(\alpha_2, \alpha_1) = (\alpha_1, \alpha_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e. $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and thus $P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Consequently, the matrix of T under the basis $\{\alpha_2, \alpha_1\}$ is

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}.$$

Definition 2. The dimension of the image space $T(V_n)$ of a linear transformation T is called the rank of T.

Conclusion :

(1) Let A be the matrix of T under a basis. Then The rank of A equals the rank of T.

(2) If the rank of T is r, then the dimension of the kernel S_T of T is $n-r$.

Example 5. Let the matrix of the linear transformation σ of 3-dimensional space V under the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ be

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Find the matrix of σ under the basis $\{\alpha_2, \alpha_3, \alpha_1\}$.

Solution. By the given condition,

$$\sigma(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, \alpha_3)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

i.e. $\begin{cases} \sigma(\alpha_1) = \alpha_1 + 4\alpha_2 + 7\alpha_3 \\ \sigma(\alpha_2) = 2\alpha_1 + 5\alpha_2 + 8\alpha_3 \\ \sigma(\alpha_3) = 3\alpha_1 + 6\alpha_2 + 9\alpha_3 \end{cases}$

and hence $\begin{cases} \sigma(\alpha_2) = 5\alpha_2 + 8\alpha_3 + 2\alpha_1 \\ \sigma(\alpha_3) = 6\alpha_2 + 9\alpha_3 + 3\alpha_1 \\ \sigma(\alpha_1) = 4\alpha_2 + 7\alpha_3 + \alpha_1 \end{cases}$

Therefore, the matrix of σ under $\{\alpha_2, \alpha_3, \alpha_1\}$ is

$$B = \begin{pmatrix} 5 & 8 & 2 \\ 6 & 9 & 3 \\ 4 & 7 & 1 \end{pmatrix}.$$