Section 3. Basis Transformation and Coordinate Transformation

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1.Basis transformation formula and transition matrix

As we know, any n linearly independent vectors can serve as a basis of an n-dimensional space V. So there are a number of bases in V. Clearly a vector has different coordinates in the distinct bases.

Question: What is the link between coordinates when a basis is transformed into another basis? In order to answer this question, let's begin with the basis transformation formula.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$ be two bases of V_n and

$$\begin{cases} \beta_1 = p_{11}\alpha_1 + p_{21}\alpha_2 + \cdots + p_{n1}\alpha_n \\ \beta_2 = p_{12}\alpha_1 + p_{22}\alpha_2 + \cdots + p_{n2}\alpha_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_n = p_{1n}\alpha_1 + p_{2n}\alpha_2 + \cdots + p_{nn}\alpha_n \end{cases} \qquad (1)$$

(1) is called the basis transformation formula between $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$.

Since $$\begin{cases} \beta_1 = p_{11}\alpha_1 + p_{21}\alpha_2 + \cdots + p_{n1}\alpha_n \\ \beta_2 = p_{12}\alpha_1 + p_{22}\alpha_2 + \cdots + p_{n2}\alpha_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_n = p_{1n}\alpha_1 + p_{2n}\alpha_2 + \cdots + p_{nn}\alpha_n \end{cases},$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = P^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

or equivalently

$$(\beta_1, \beta_2, \cdots, \beta_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)P$$

The basis transformation formula

In this formula, P is called the transition matrix from $\alpha_1, \alpha_2, \cdots, \alpha_n$ to $\beta_1, \beta_2, \cdots, \beta_n$.

It can be checked that the transition matrix P is invertible

2.Coordinate transformation formula

Theorem 1. Let α be an element of V_n. If the coordinates of α under the bases $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$ are $(x'_1, x'_2, \cdots, x'_n)^T$ and $(x_1, x'_2, \cdots, x'_n)^T$ respectively, and the two bases satisfy

$$(\beta_1, \beta_2, \cdots, \beta_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)\boldsymbol{P}$$

then we have

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Proof.

Since $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=(\beta_1,\beta_2,\cdots,\beta_n)\begin{pmatrix}x_1'\\x_2'\\\vdots\\x_n'\end{pmatrix}$

$$(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)P,$$

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=(\alpha_1,\alpha_2,\cdots,\alpha_n)P\begin{pmatrix}x_1'\\x_2'\\\vdots\\x_n'\end{pmatrix}.$$

namely, $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}.$

Since P is invertible,

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example 1. In $P[x]_3$, let

$$\alpha_1 = x^3 + 2x^2 - x,\ \alpha_2 = x^3 - x^2 + x + 1,$$

$$\alpha_3 = -x^3 + 2x^2 + x + 1,\ \alpha_4 = -x^3 - x^2 + 1,$$

and $\beta_1 = 2x^3 + x^2 + 1,\qquad \beta_2 = x^2 + 2x + 2,$

$$\beta_3 = -2x^3 + x^2 + x + 2,\quad \beta_4 = x^3 + 3x^2 + x + 2.$$

Find the coordinte transformation formula.

Solution. First, use $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ to express $\beta_1,\beta_2,\beta_3,\beta_4$.

Since $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (x^3, x^2, x, 1)A,$

$$(\beta_1,\beta_2,\beta_3,\beta_4) = (x^3, x^2, x, 1)B,$$

$$\textit{where}\quad A=\begin{pmatrix}1 & 1 & -1 & -1\\ 2 & -1 & 2 & -1\\ -1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\end{pmatrix}, B=\begin{pmatrix}2 & 0 & -2 & 1\\ 1 & 1 & 1 & 3\\ 0 & 2 & 1 & 1\\ 1 & 2 & 2 & 2\end{pmatrix},$$

we have $(\beta_1,\beta_2,\beta_3,\beta_4)=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)A^{-1}B.$

Hence the coordinate transformation formula is

$$\begin{pmatrix}x_1'\\ x_2'\\ x_3'\\ x_4'\end{pmatrix}=B^{-1}A\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}.$$

What is left is the computation of $B^{-1}A$.

$$(B \mid A) = \left(\begin{array}{cccc|cccc} 2 & 0 & -2 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 3 & 2 & -1 & 2 & -1 \\ 0 & 2 & 1 & 1 & -1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 \end{array}\right)$$

Elementary row operations

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1 \end{array}\right) = \left(E \mid B^{-1}A\right)$$

Hence

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Example 2. Geometric explanation of coordinate transformation.

Let $\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $\beta_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

be two bases of $V = R^2$.

Then the coordinate of $\alpha = -\frac{1}{2}\alpha_1 + \alpha_2$ under α_1, α_2 is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

By the coordinate transformation formula, the coordianate of α under β_1, β_2 is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1/2 \end{pmatrix}^{-1} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$$

i.e. $\alpha = \frac{1}{2}\beta_1 - \beta_2.$