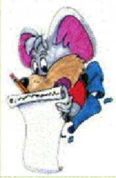
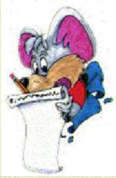
Section 2. Dimension, Bases and Coordinates



1. Dimension and bases of a linear space

As we know, there are at most n linearly independent vectors in R^n. In a general linear space, there exits a similar conclusion. In this section, we shall investigate this topic. Let's begin with the dimension and bases of a linear space.

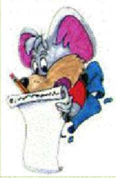


Definition 1. If $\alpha_1, \alpha_2, \cdots, \alpha_n \in V$ satisfy the following conditions, then $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is called a basis of V and n the dimension of V.

(1) $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independen t.

(2) Any other element in V can be linearly expressed by $\alpha_1, \alpha_2, \cdots, \alpha_n$.

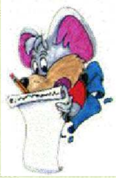
A linear space V with dimension n is called an n-dimensional linear space, which is denoted by V_n.



If a linear space has infinite linearly independent Vectors, then this space is said to be infinite-dimensional.

Clearly, if $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a basis of V_n then V_n can be written in the form

$$V_n = \{\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n \mid x_1, x_2, \cdots, x_n \in R\}$$



2. Coordinate

Definition 2. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a basis of V_n, so for any $\alpha \in V$, there exist $x_1, x_2, \cdots, x_n$ such that $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$. We shall say that $(x_1, x_2, \cdots, x_n)^T$ is the coordinate under the basis $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

It coordinate of a vector is unique if the basis is fixed.

Example 1. In $P[x]_5$, let $p_1 = 1, p_2 = x, p_3 = x^2, p_4 = x^3$ $p_5 = x^4$. Then $\{p_1, p_2, p_3, p_4, p_5\}$ is a basis of $P[x]_4$ since every polynomial p of degree less than 5 can be expressed as

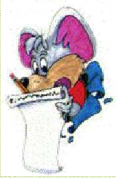
$$p = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

which is

$$p = a_0p_1 + a_1p_2 + a_2p_3 + a_3p_4 + a_4p_5$$

Hence, the coordinate of p under $\{p_1, p_2, p_3, p_4, p_5\}$ is

$$(a_0, a_1, a_2, a_3, a_4)^T.$$

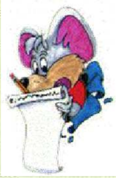


If the basis $q_1 = 1, q_2 = 1 + x, q_3 = 2x^2, q_4 = x^3, q_5 = x^4$ is selected, then

$$p = (a_0 - a_1)q_1 + a_1 q_2 + \frac{1}{2} a_2 q_3 + a_3 q_4 + a_4 q_5$$

The coordinate of p becomes

$$(a_0 - a_1, a_1, 1/2a_2, a_3, a_4)^T.$$



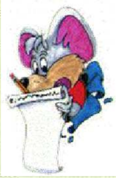
Example 2. The set of all real matrix of order 2 is a linear space for the addition of matrices and Multiplication of matrix by a number. In this space

$$\text{let } E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we have

$$k_1 E_{11} + k_2 E_{12} + k_3 E_{21} + k_4 E_{22} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$



Hence,

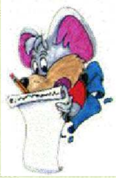
$$k_1 E_{11} + k_2 E_{12} + k_3 E_{21} + k_4 E_{22} = O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Leftrightarrow \qquad k_1 = k_2 = k_3 = k_3 = 0,$$

namely $E_{11}, E_{12}, E_{21}, E_{22}$ are linearly independent.

For an arbitrary real matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in V,$$



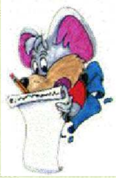
we have

$$A = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$$

whence $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis of V.

The coordinate of A under this basis is

$$(a_{11}, a_{12}, a_{21}, a_{22}).$$



Example 3. In $R[x]_n$, let

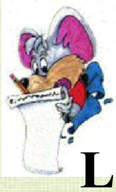
$$\varepsilon_1 = 1, \varepsilon_2 = (x-a), \varepsilon_3 = (x-a)^2, \cdots, \varepsilon_n = (x-a)^{n-1}$$

By Taylor Formula,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$

So the coordinate of $f(x)$ **under the basis** $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_n\}$ **is**

$$\left(f(a), f'(a), \frac{f''(a)}{2!}, \cdots, \frac{f^{(n-1)}(a)}{(n-1)!}\right)^T.$$



3. Isomorphism between linear spaces

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be a basis of n – dimensional space V_n. Under this basis, each vector of V_n has a uniquely determined coordinate which belongs to R^n. To each vector corresponds its coordinate so that we obtain a mapping from V_n to R^n.

It is worth noticing that this mapping demonstrates very good behaviors. First, the mapping is surjective and injective, or equivalently a one - one mapping. Next, the mapping is operation - perserving. We shall discuss such mapping in the name of isomophism.

Definition. Let U,V be two linear space. If there exists a one-one correspondence between them and the correspondence preserves the linear operations, then U and V are called isomorphic.

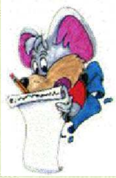
For example. An $n-$dimensional linear space V_n and R^n are isomorphic.

Indeed, let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be a basis of V_n. Then

$$V_n=\{\alpha=x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n \mid x_1,x_2,\cdots,x_n\in R\}$$

Let $\alpha=x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n$

$$\leftrightarrow (x_1,x_2,\cdots,x_n)^T.$$



Clearly, this correspondence is one-one.

In addition, let

$$\alpha \leftrightarrow (x_1, x_2, \cdots, x_n)^T \quad \beta \leftrightarrow (y_1, y_2, \cdots, y_n)^T$$

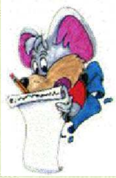
i.e. $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$

$$\beta = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n$$

Then

$$\alpha + \beta = (x_1 + y_1)\alpha_1 + (x_2 + y_2)\alpha_2 + \cdots + (x_n + y_n)\alpha_n$$

$$k\alpha = kx_1\alpha_1 + kx_2\alpha_2 + \cdots + kx_n\alpha_n$$

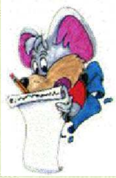


That is to say,

$$\alpha + \beta \leftrightarrow (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)^T$$
$$= (x_1, x_2, \cdots, x_n)^T + (y_1, y_2, \cdots, y_n)^T$$

$$\alpha + \beta \leftrightarrow (kx_1, kx_2, \cdots, kx_n)^T$$
$$= k(x_1, x_2, \cdots, x_n)^T$$

Hence the correspondence preserves the linear operations, and thus *U* and *V* are isomorphic.

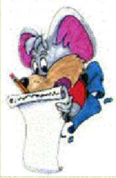


Properties of isomorphism

1. Isomorphism is reflexive, symmetric and transitive.

2. Any two *n*-dimensional linear spaces are isomorphic.

For a linear space, there are two essential elements, a set and two operations. If there is a one-one correspondence between sets and the two operations between the linear spaces are preserved. Then these two linear spaces are identical in essence. Therefore,



we shall differentiate two isomorphic linear space in mathematics.