Section 1 The Definition and Properties of Linear Space

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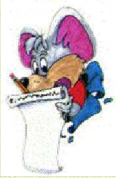
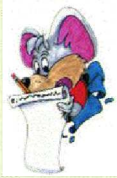


1. The definition of linear space

Definition 1. A linear space V is a set of objects which can be added and multiplied by numbers, in such a way that the sum of two elements of V is again an element of V, and the product of an element of V by a number is an element of V, and the following properties are satisfied for any given $\alpha, \beta, \gamma \in V$ and any real numbers λ, μ.

(1) $\alpha + \beta = \beta + \alpha.$



(2) $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$.

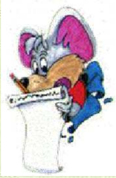
(3) There is an element of V, denoted by 0, such that $\alpha+0=\alpha$ for all elements α of V.

(4) Given any element α of V, there is an element β, called the negative element of α, such that

$$\alpha+\beta=0.$$

(5) $1\alpha=\alpha$.

(6) $\lambda(\mu\alpha)=(\lambda\mu)\alpha$.



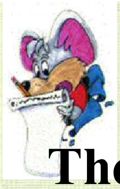
(7) $(\lambda + \mu)\alpha = \lambda\alpha + \mu\alpha$.

(8) $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$.

Remarks:

1. The addition and multiplication by a number in a linear space are called linear operations.

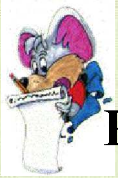
2 . A linear space is also called a vector space and an element is accordingly called a vector. However, a vector is not a set of ordered numbers any more.



The approach to checking whether a set is a linear space

(1) In a set, if the involved addition and multiplication are just operations of real numbers, then what is needed is to check the closedness of operations.

Example 1. Denote the set of all $m \times n$ matrices by $R^{m\times n}$. Show that $R^{m\times n}$ forms a linear space for the addition of matrices and multiplication of matrix by a number.



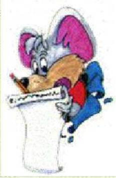
Proof. $A_{m\times n}+B_{m\times n}=C_{m\times n},\lambda A_{m\times n}=D_{m\times n},$

i.e. $R^{m\times n}$ is closed for the addition of matrices and multiplication of matrix by a number, thus a linear space.

Example 2. Denote the set of all polynomials of degree less than n by $P[x]_n$, i.e.

$$P[x]_n=\{p(x)=a_nx^n+\cdots+a_1x+a_0,a_0,\cdots,a_n\in R\},$$

Show that $P[x]_n$ forms a linear space for the addition of polynomials and multiplication of polynomial by a number.



Proof.

$$(a_n x^n + \cdots + a_1 x + a_0) + (b_n x^n + \cdots + b_1 x + b_0)$$

$$= (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0) \in P[x]_n$$

$$\lambda(a_n x^n + \cdots + a_1 x + a_0)$$

$$= (\lambda a_n) x^n + \cdots + (\lambda a_1) x + (\lambda a_0) \in P[x]_n$$

$P[x]_n$ is closed for both operations, and thus a linear space.

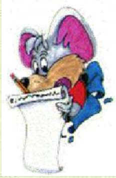
Example 3. Denote the set of all polynomials of degree n by $Q[x]_n$, i.e.

$$Q[x]_n = \{ p(x) = a_n x^n + \cdots + a_1 x + a_0 \mid a_0, \cdots, a_n \in R, \quad a_n \neq 0 \}.$$

In this case, since

$$0p = 0x^n + \cdots + 0x + 0 \notin Q[x]_n$$

$Q[x]_n$ is not closed for the multiplication of polynomial by a number, and thus not a linear space.



Example 4. The set

$$S[x] = \{ s = A\sin(x+B) \mid A, B \in R \}.$$

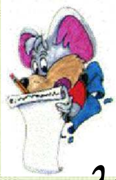
forms a linear space for the addition of functions multiplication of function by a number.

Proof. $s_1 + s_2 = A_1\, sin(x + B_1) + A_2\, sin(x + B_2)$

$$= (a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x)$$

$$= (a_1 + a_2)\cos x + (b_1 + b_2)\sin x$$

$$= A\sin(x + B) \in S[x].$$



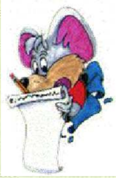
$$\lambda s_1 = \lambda A_1 \sin(x + B_1) = (\lambda A_1)\sin(x + B_1) \in S[x]$$

Hence, $S[x]$ is a linear space.

Example 5. The set of all real continuous functions on [*a,b*] forms a linear space for the addition of functions and multiplication of function by a number.

The proof is left to the reader.

(2) Generally, the eight properties should be verified one by one in order to judge whether a set forms a linear space.



Example 6.

The set R^+ of positive real numbers is a linear space for the addition and multiplication by a number defined below.

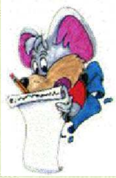
$$a \oplus b = ab, \quad \lambda \circ a = a^{\lambda}, (\lambda \in R, a, b \in R^+).$$

Proof.

$$\forall a, b \in R^+, \Rightarrow a \oplus b = ab \in R^+;$$

$$\forall \lambda \in R, a \in R^+, \Rightarrow \lambda \circ a = a^{\lambda} \in R^+.$$

These two operations are closed.



The eight properties are verified one by one in the following:

(1) $a \oplus b = ab = ba = b \oplus a$;

(2) $(a \oplus b) \oplus c = (ab) \oplus c = (ab)c = a \oplus (b \oplus c)$;

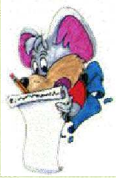
(3) The zero element is 1 in R^+. Indeed,

for all $a \in R^+$, $a \oplus 1 = a \cdot 1 = a$;

(4) $\forall a \in R^+$, its negative element is $a^{-1} \in R^+$.

Indeed,

$$a \oplus a^{-1} = a \cdot a^{-1} = 1;$$



(5) $1 \circ a = a^1 = a;$

(6) $\lambda \circ (\mu \circ a) = \lambda \circ a^{\mu} = (a^{\mu})^{\lambda} = a^{\lambda\mu} = (\lambda\mu) \circ a;$

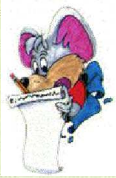
(7) $(\lambda + \mu) \circ a = a^{\lambda+\mu} = a^{\lambda} a^{\mu} = a^{\lambda} \oplus a^{\mu}$

$= \lambda \circ a \oplus \mu \circ a;$

(8) $\lambda \circ (a \oplus b) = \lambda \circ (ab) = (ab)^{\lambda} = a^{\lambda} b^{\lambda}$

$= a^{\lambda} \oplus b^{\lambda} = \lambda \circ a \oplus \lambda \circ b.$

Hence, R^+ is a linear space.



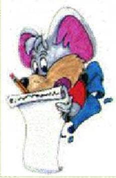
Example 7. The set of *n* ordered real numbers

$$S^n = \left\{ x = (x_1, x_2, \cdots, x_n)^T \mid x_1, x_2, \cdots, x_n \in R \right\}$$

is not a linear space for the addition of vectors and multiplication by a number defined below

$$\lambda \circ (x_1, \cdots, x_n)^T = (0, \cdots, 0)$$

due to $1 \circ x = (0,0,\cdots,0)\ (\forall x \in S^n)$.



2. Properties of linear spaces

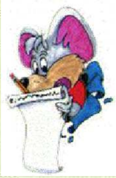
Property 1. The zero element is unique.

Proof. Suppose there exist two zero elements $0_1, 0_2$.

Then for any $\alpha \in V$, we have $\alpha + 0_1 = \alpha$, $\alpha + 0_2 = \alpha$.

Since $0_1, 0_2 \in V$, $0_2 + 0_1 = 0_2, 0_1 + 0_2 = 0_1$.

Hence, $0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$.



Property 2. **The negative element of any element is unique.**

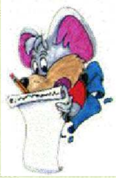
Proof. **Suppose that** α **has two negative elements** β, γ**.**

Then $\alpha + \beta = 0, \quad \alpha + \gamma = 0.$

Thus

$$\begin{aligned} \beta = \beta + 0 &= \beta + (\alpha + \gamma) \\ &= (\beta + \alpha) + \gamma \\ &= 0 + \gamma = \gamma. \end{aligned}$$

The negative element of α **will be denoted by** $-\alpha$**.**



3. $0\alpha = 0$; $(-1)\alpha = -\alpha$; $\lambda 0 = 0$.

Proof. $\alpha + 0\alpha = 1\alpha + 0\alpha = (1+0)\alpha = 1\alpha = \alpha$,

i.e. $0\alpha = 0$.

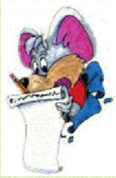
From $\alpha + (-1)\alpha = 1\alpha + (-1)\alpha = [1+(-1)]\alpha = 0\alpha = 0$,

it follows that $(-1)\alpha = -\alpha$.

$$\lambda 0 = \lambda[\alpha + (-1)\alpha] = \lambda\alpha + (-\lambda)\alpha$$

$$= [\lambda + (-\lambda)]\alpha = 0\alpha$$

$$= 0.$$

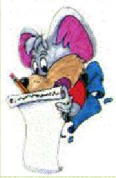


Property 4. $\lambda\alpha = 0$ **implies that** $\lambda = 0$ **or** $\alpha = 0$**.**

Proof. **If** $\lambda \neq 0$, **then** $\frac{1}{\lambda}(\lambda\alpha) = \frac{1}{\lambda} \cdot 0 = 0.$

Since $\frac{1}{\lambda}(\lambda\alpha) = \frac{1}{\lambda} \cdot \lambda \cdot \alpha = \alpha,$

$\alpha = 0.$



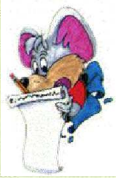
3. Subspaces

Definition 2. Let V be a linear space and L a non-empty subset of V. If L itself is a linear space for the addition and multiplication by a number in V, then L is called a subspace of V .

Theorem. A subset of linear space V is a subspace of V iff L is closed for the linear operations in V.

Example 8.

Whether are the following subsets of $R^{2\times 3}$ linear space? Why?



$$(1)\ W_1=\left\{\begin{pmatrix}1 & b & 0\\ 0 & c & d\end{pmatrix}\middle|\, b,c,d\in R\right\};$$

$$(2)\ W_2=\left\{\begin{pmatrix}a & b & 0\\ 0 & 0 & c\end{pmatrix}\middle|\, a+b+c=0, a,b,c\in R\right\}.$$

Solution.

(1) is not a subspace since

$$A=B=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\in W_1$$

and $$A+B=\begin{pmatrix}2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\notin W_1.$$

W_1 **is not closed for the addition, and thus not a subspace.**

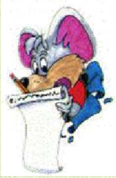
(2) Since $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in W_2$, W_2 **is non-empty.**

Moreover, if

$$A = \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix} \in W_2$$

then $a_1 + b_1 + c_1 = 0, \quad a_2 + b_2 + c_2 = 0,$

i.e. $A + B = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 & 0 \\ 0 & 0 & c_1 + c_2 \end{pmatrix}$



and $$(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = 0,$$

Hence $A + B \in W_2$. In addition, for all $k \in R$

$$kA = \begin{pmatrix} ka_1 & kb_1 & 0 \\ 0 & 0 & kc_1 \end{pmatrix}$$

and $$ka_1 + kb_1 + kc_1 = 0,$$

i.e. $kA \in W_2$.

Therefore, W_2 is a subspace of $R^{2\times 3}$.