Section 8 Positive Definite Quadratic Form

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1. Inertial theorem

A real quadratic form can be transformed into canonical form through different methods. Although canonical form is not unique, the terms contained in the canonical form is definite which equals the rank of the quadratic form.

In the sequel, we investigate properties of canonical form of real quadratic forms. We assume that the involved linear substitutions are all real.

Theorem 1 (*Inertial theorem*).

Let $f = x^T Ax$ be a real quadratic form whose rank is r. If both $x = Cy$ and $x = Pz$ transform it into

$$f = k_1 y_1^2 + k_2 y_2^2 + \cdots + k_r y_r^2 \quad (k_i \neq 0),$$

and

$$f = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \cdots + \lambda_r z_r^2 \quad (\lambda_i \neq 0)$$

respectively, then the numbers of positive numbers in $k_1, \cdots, k_r$ and $\lambda_1, \cdots, \lambda_r$ are equal.

2. The concept of positive and negative definite quadratic form

Definition 1. Let $f(x) = x^T Ax$. If $f(x) > 0$ ($f(x) < 0$) holds for any $x \neq 0$, then f is called a positive (negative) definite quadratic form and the matrix A a positive (negative) definite matrix.

Example.

$f = x^2 + 4y^2 + 16z^2$ is positive definite.

$f = -x_1^2 - 3x_2^2$ is negative definite.

3. Identifying positive (negative) definite quadratic forms

Theorem 2 A real quadratic form $f = x^T Ax$ is positive definite iff all the n coefficients of its canonical form are positive.

Proof. Let the invertible substitution $x = Cy$

$$f(x) = f(Cy) = \sum_{i=1}^{n} k_i y_i^2 .$$

Sufficiency.

If $k_i > 0\ (i = 1, \cdots, n)$, then

$$y = C^{-1}x \neq 0 \text{ holds for } x \neq 0.$$

Hence $f(x) = \sum_{i=1}^{n} k_i y_i^2 > 0.$

Necessity.

Assume that $k_s \le 0$. By letting $y = e_s$, we have

$$f\left(Ce_s\right) = k_s \le 0.$$

Clearly, $Ce_s \ne 0$ which contradicts positive definiteness of f. Therefore, $k_i > 0\left(i = 1, \cdots, n\right)$.

Corollary A symmetric matrix is positive definite iff all its eigenvalues are positive.

Theorem 3 A symmetric matrix A is positive definite iff all its order principal minors are positive, i.e.

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0;$$

A symmetric matrix A is negative definite iff all its principal minors of odd orders are negative all its order principal minors of even orders are positive, i.e.

$$(-1)^r \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix} > 0, \quad (r = 1, 2, \cdots, n).$$

The above theorem is called Hurwitz Theorem.

Properties of positive definite matrices:

1. If A is positive definite, then A^T, A^{-1}, A^* are all positive definite.

2. If both A and B are positive definite matrices of order n, then $A+B$ is positive definite.

Example 1. Check whether the quadratic form

$$f(x_1,x_2,x_3)=5x_1^2+x_2^2+5x_3^2+4x_1x_2-8x_1x_3-4x_2x_3$$

is positive definite or not.

Solution. The matrix of f is $\begin{pmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{pmatrix}$,

whose order principal minors are respectively

$$5>0, \quad \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}=1>0, \quad \begin{pmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{pmatrix}=1>0,$$

Consequently, the matrix as well as f is positive definite.

Example. **Check whether the quadratic form**

$$f(x_1, x_2, x_3) = 2x_1^2 + 4x_2^2 + 5x_3^2 - 4x_1x_3$$

is positive definite or not.

Solution. **The matrix of f is**

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{pmatrix},$$

From $|\lambda E - A| = 0$, it follow that

$$\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6.$$

Hence A is a positive definite matrix and thus the corresponding quadratic form is positive definite.

Example 3. Judge whether

$$f = -5x^2 - 6y^2 - 4z^2 + 4xy + 4xz$$

is definite or not.

Solution. **The matrix of f is**

$$A = \begin{pmatrix} -5 & 2 & 2 \\ 2 & -6 & 0 \\ 2 & 0 & -4 \end{pmatrix},$$

$$a_{11} = -5 < 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} -5 & 2 \\ 2 & -6 \end{vmatrix} = 26 > 0,$$

$$|A| = -80 < 0.$$ **Hence f is negative definite.**