# Section 7. Completing the Square to Transform a Quadratic Form into Canonical Form

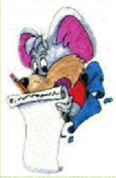
7

## 1. Lagrange's method of completing the square

In the previous section, we transform a quadratic form into a canonical form.

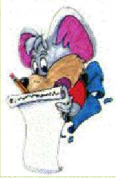
Question: Is there any other method to transform a quadratic form into canonical form.

In this section, we shall introduce an alternative Method--the Lagrange's method of completing the square.



## Lagrange's method is as follows.

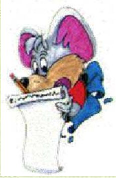
1. If $f$ contains $x_i^2$ for some $i$, then put $x_i$ – related terms together. Afterwards, complete the square. Continue in this way until all the variables become squared. After a succession of non-degenerate linear substitutions, a canonical form is available.



**2. If $f$ does not contain any $x_i^2$ and there exists $a_{ij} \neq 0$ $(i \neq j)$, then make the linear substitution**

$$\begin{cases} x_i = y_i - y_j \\ x_j = y_i + y_j \\ \quad x_k = y_k \end{cases} \quad (k = 1,2,\cdots,n \text{ and } k \neq i, j).$$

**We obtain a quadratic form with some square terms, follow the above procedure to complete the square.**



**Example 1** Transform the quadratic form

$$f = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3$$

into a canonical form, and find out the transformation matrix.

**Solution.**

Square term

$x_1$ − related term

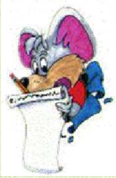
$$f = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3$$

$$= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 5x_3^2 + 6x_2x_3$$

$$= (x_1 + x_2 + x_3)^2$$

Subtract redundant terms

$$- x_2^2 - x_3^2 - 2x_2x_3 + 2x_2^2 + 5x_3^2 + 6x_2x_3$$

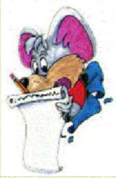


$$=\left(x_1+x_2+x_3\right)^2+x_2^2+4x_3^2+4x_2x_3$$

$$=\left(x_1+x_2+x_3\right)^2+\left(x_2+2x_3\right)^2.$$

Let $\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = x_2 + 2x_3 \\ y_3 = x_3 \end{cases}$ Then $\begin{cases} x_1 = y_1 - y_2 + y_3 \\ x_2 = y_2 - 2y_3 \\ x_3 = y_3 \end{cases}$

namely $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

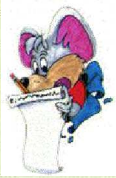


Hence $f = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3$

$$= y_1^2 + y_2^2.$$

The transformation matrix is

$$C = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(|C| = 1 \neq 0\right).$$



**Example 2.**

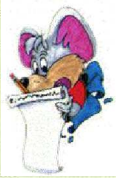
**Transform** $f = 2x_1x_2 + 2x_1x_3 - 6x_2x_3$ **into a canonical form and find out the transformation matrix.**

**Solution.** **Since** $f$ **does not contain any square term**

**Let** $\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2, \\ x_3 = y_3 \end{cases}$ $\left( \text{or} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right)$

**Substituting into** $f = 2x_1x_2 + 2x_1x_3 - 6x_2x_3$ **yields**

$$f = 2y_1^2 - 2y_2^2 - 4y_1y_3 + 8y_2y_3.$$



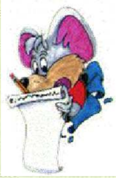
Completing the square, we have

$$f = 2(y_1 - y_3)^2 - 2(y_2 - 2y_3)^2 + 6y_3^2.$$

Let $\begin{cases} z_1 = y_1 - y_3 \\ z_2 = y_2 - 2y_3 \\ z_3 = y_3 \end{cases}$

Then $\begin{cases} y_1 = z_1 + z_3 \\ y_2 = z_2 + 2z_3, \\ y_3 = z_3 \end{cases}$ $\left( \text{or} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right)$

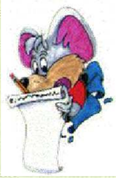
So $f = 2z_1^2 - 2z_2^2 + 6z_3^2.$



**To make a summation, the transformation matrix is**

$$C=\begin{pmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 1 & 3\\ 1 & -1 & -1\\ 0 & 0 & 1\end{pmatrix}. \quad \left(|C|=-2\neq 0\right).$$



The desired canonical form is

$$f = z_1^2 - z_2^2 - z_3^2,$$

The linear substitution is

$$\begin{cases} x_1 = z_1 - z_2 - z_3, \\ x_2 = z_1 + z_2 - z_3, \\ x_3 = z_3. \end{cases}$$