Section 6 Canonical Form of a Quadratic Form

1. The concept of quadratic form and canonical form

In the quadratic form

$$f(x_1, x_2, \cdots, x_n) = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$$
$$+ 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots + 2a_{n-1,n}x_{n-1}x_n$$

if a_{ij} $(i,j = 1,2,\cdots,n)$ **are all complex numbers,**

f **is called a complex quadratic form;**

if a_{ij} $(i,j = 1,2,\cdots,n)$ **are all real numbers,**

f **is called a real quadratic form.**

2. Transform a quadratic form into canonical form

Our aim is to seek an invertible linear substitution to transform a quadratic form into a canonical form.

Let

$$\begin{cases} x_1 = c_{11}y_1 + c_{12}y_2 + \cdots + c_{1n}y_n, \\ x_2 = c_{21}y_1 + c_{22}y_2 + \cdots + c_{2n}y_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = c_{n1}y_1 + c_{n2}y_2 + \cdots + c_{nn}y_n \end{cases}$$

Write $C = (c_{ij})$.

The above linear substitution is expressed by

$$x = Cy$$

Substituting into $f = x^T Ax$, we have

$$f = x^T Ax = (Cy)^T A(Cy) = y^T\left(C^T AC\right)y.$$

Theorem 1. **Let C be invertible and $B = C^T AC$. If A is symmetric, then B is symmitric and $R(B) = R(A)$.**

Proof. **Since $A = A^T$,**

$$B^T = \left(C^T AC\right)^T = C^T A^T C = C^T AC = B,$$

i.e. B is symmetric.

From $B = C^T AC$ it follows that $R(B) \le R(AC) \le R(A)$.

Moreover $A = \left(C^T\right)^{-1} BC^{-1}$, so $R(A) \le R\left(BC^{-1}\right) \le R(B)$.

Consequently, $R(A) = R(B)$.

Remark.

1. After an invertible linear substituti on $x = Cy$, the rank of f remains unchanged , but its matrix becomes $B = C^T AC$.

2. Transformi ng f into canonical form through the linear substituti on $x = Cy$ amounts to making f become

$$y^T C^T ACy = k_1 y_1^2 + k_2 y_2^2 + \cdots + k_n y_n^2$$

$$= (y_1, y_2, \cdots, y_n) \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

which implies that $C^T AC$ is a diagonal matrix .

Since A is a real symmetric matrix , there exists P such that $P^{-1}AP = \Lambda$, namely $P^{T}AP = \Lambda$. Hence we have the following result.

Theorem 2. For every quadratic form $f = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ $(a_{ij} = a_{ji})$, there exists an orthogonal substituti on $x = Py$ which makes f become canonocal form

$$f = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A = (a_{ij})$.

The steps to transform a quadratic form into canonical form are as follows.

1. Find A so that f can be written in the form

$$f = x^T A x.$$

2. Find the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A.

3. Find correspond ing eigenvecto rs $\xi_1, \xi_2, \cdots, \xi_n$.

4. Orthonorma lize $\xi_1, \xi_2, \cdots, \xi_n$ to obtain $\eta_1, \eta_2, \cdots, \eta_n$ and write $C = (\eta_1, \eta_2, \cdots, \eta_n)$.

5. Make the orthogonal substituti on $x = Cy$ to transform f into the canonical form

$$f = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.$$

Example 2.

Use an orthogonal substitution to transform

$$f = 17x_1^2 + 14x_2^2 + 14x_3^2 - 4x_1x_2 - 4x_1x_3 - 8x_2x_3$$

into a canonical form.

Solution. 1. Write down the matrix of f and find its eigenvalues.

$$A = \begin{pmatrix} 17 & -2 & -2 \\ -2 & 14 & -4 \\ -2 & -4 & 14 \end{pmatrix}$$

$$|A - \lambda E| = \begin{pmatrix} 17-\lambda & -2 & -2 \\ -2 & 14-\lambda & -4 \\ -2 & -4 & 14-\lambda \end{pmatrix} = (\lambda - 18)^2(\lambda - 9)$$

The eigenvalues are $\lambda_1 = 9, \lambda_2 = \lambda_3 = 18.$

2. Find eigenvectors.

Replacing λ by 9 in $(A-\lambda E)x=0$, we obtain a system of fundamental solutions $\xi_1 = (1/2,1,1)^T$.

Similarly, replacing λ by 18 in $(A-\lambda E)x=0$, we have

$$\xi_2 = (-2,1,0)^T,\ \xi_3 = (-2,0,1)^T.$$

3. Orthogonal ize ξ_1, ξ_2, ξ_3.

Let $\alpha_1 = \xi_1, \alpha_2 = \xi_2,\ \alpha_3 = \xi_3 - \dfrac{[\alpha_2, \xi_3]}{[\alpha_2, \alpha_2]}\alpha_2.$

We obtain the orthogonal vectors

$$\alpha_1 = (1/2,1,1)^T,\quad \alpha_2 = (-2,1,0)^T,$$

$$\alpha_3 = (-2/5,-4/5,1)^T.$$

4. Normalize these orthogonal vectors.

By Letting $\eta_i = \dfrac{\alpha_i}{\|\alpha_i\|}, \quad (i = 1,2,3),$

we have $\eta_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \eta_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} -2/\sqrt{45} \\ -4/\sqrt{45} \\ 5/\sqrt{45} \end{pmatrix}.$

Write $P = \begin{pmatrix} 1/3 & -2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{pmatrix}.$

Use P to construct the orthogonal substitution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/3 & -2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

Then $f = 9y_1^2 + 18y_2^2 + 18y_3^2.$

Example 3. Find an orthogonal substitution $x = Py$ to transform

$$f = 2x_1x_2 + 2x_1x_3 - 2x_1x_4 - 2x_2x_3 + 2x_2x_4 + 2x_3x_4$$

into a canonical form.

Solution.

The matrix of f is

$$A = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix},$$

whose characteristic polynomial is

$$|A-\lambda E| = \begin{vmatrix} -\lambda & 1 & 1 & -1 \\ 1 & -\lambda & -1 & 1 \\ 1 & -1 & -\lambda & 1 \\ -1 & 1 & 1 & -\lambda \end{vmatrix}.$$

Adding the 2nd, 3rd, 4th column to the 1st column yields

$$|A-\lambda E| = (-\lambda+1)\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & -\lambda & -1 & 1 \\ 1 & -1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix},$$

Subtracting the row 1 from 2,3, 4 respectively leads to

$$|A-\lambda E| = (-\lambda+1)\begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & -\lambda-1 & -2 & 2 \\ 0 & -2 & -\lambda-1 & 2 \\ 0 & 0 & 0 & -\lambda+1 \end{vmatrix}$$

$$= (-\lambda+1)^2\begin{vmatrix} -\lambda-1 & -2 \\ -2 & -\lambda-1 \end{vmatrix}$$

$$= (-\lambda+1)^2(\lambda^2+2\lambda-3) = (\lambda+3)(\lambda-1)^3.$$

Thus eigenvalues of A are $\lambda_1 = -3, \lambda_2 = \lambda_3 = \lambda_4 = 1.$

For $\lambda_1 = -3$, solve the system $(A+3E)x = 0.$

A system of fundamental solutions is $\xi_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, Which is normalized as $p_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = \lambda_3 = \lambda_4 = 1$, solve the system $(A - E)x = 0$

to obtain a system of fundamental solutions

$$\xi_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \xi_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

Normalize them $p_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, p_3 = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, p_4 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$

The desired orthogonal substitution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 0 & 1/2 \\ -1/2 & 1/\sqrt{2} & 0 & -1/2 \\ -1/2 & 0 & 1/\sqrt{2} & 1/2 \\ 1/2 & 0 & 1/\sqrt{2} & -1/2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

and $f = -3y_1^2 + y_2^2 + y_3^2 + y_4^2.$