Section 5 The Expressions of and rank of a Quadratic Form

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1. Expressions of quadratic form

1. Expression by the use of summation symbol

A quadratic form is the expression

$$f(x_1, x_2, \cdots, x_n) = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$$
$$+ 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots + 2a_{n-1,n}x_{n-1}x_n$$

By putting $a_{ji} = a_{ij}$, $2a_{ij}x_ix_j = a_{ij}x_ix_j + a_{ji}x_jx_i$. Thus

$$\begin{aligned} f &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ &+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ &+ \cdots + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 \\ &= \sum_{i,j=1}^{n} a_{ij}x_ix_j. \end{aligned}$$

2. Expressed in the form of matrix

$$
\begin{aligned}
f &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\
&+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\
&+ \cdots + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 \\
&= x_1(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) \\
&+ x_2(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) \\
&+ \cdots + x_n(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) \\
&= (x_1, x_2, \cdots, x_n)\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}
\end{aligned}
$$

$$= (x_1, x_2, \cdots, x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Write $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$

Then a quadratic form can be written as $f = x^T A x$**,**
where A **is a symmetric matrix.**

2. The rank of a quadratic form

As we know, given a quadratic form a symmetric matrix is uniquely determined. Conversely， given a symmetric matrix， it is clear that a quadratic form can be determined. So there is one-one correspondence between quadratic forms and symmetric matrices.

A symmetric matrix A is called the matrix of the quadratic form f provided that $f = x^T Ax$.

f is referred to as the quadratic form of A.

The rank of A is accordingly called the rank of f.

Example 1 .

Find the matrix of the quadratic form

$$f = x_1^2 + 2x_2^2 - 3x_3^2 + 4x_1x_2 - 6x_2x_3.$$

Solution. $a_{11} = 1, a_{22} = 2, a_{33} = -3,$

$a_{12} = a_{21} = 2, \quad a_{13} = a_{31} = 0,$

$a_{23} = a_{32} = -3.$

Hence $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & -3 \\ 0 & -3 & -3 \end{pmatrix}.$