Section 3 Similar Matrices

3

1. The concept of similar matrices and similar transformations

Definition 1. Let A, B be square matices of order n. If there exists an invertible matrix P such that

$$P^{-1}AP = B,$$

then A is said to be similar to B. If A is performed the operation $P^{-1}AP$, then we say that a similar transformation is carried out on A. The invertible matrix P is called the similarity tranformation matrix.

2. Properties of similar matrices and similar transformations

1. Similarity relation is an equivalency relation.

(1) Reflexivity : A is similar to A.

(2) Symmetry : If A is similar to B, then B is similar to A.

(3) Transtivity :

If A is similar to B, and B is similar to C, then A is similar to C.

2. $P^{-1}(A_1 A_2)P = (P^{-1} A_1 P)(P^{-1} A_2 P)$.

3. If A and B are similar, then A^m and B^m are similar (m is an arbitrary positive integer).

4. $P^{-1}(k_1A_1 + k_2A_2)P = k_1P^{-1}A_1P + k_2P^{-1}A_2P$

where k_1, k_2 are arbitrary constants .

Theorem 1. If A and B are similar, then A and B have the same chracteristic polynomial, and thus the same eigenvalues.

Proof. Since A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$.

Hence
$$\begin{aligned}|B - \lambda E| &= |P^{-1}AP - P^{-1}(\lambda E)P| \\ &= |P^{-1}(A - \lambda E)P| \\ &= |P^{-1}||A - \lambda E||P| \\ &= |A - \lambda E|.\end{aligned}$$

Corollary. **If A is similar to the diagonal matrix**

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

then $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the n eigenvalues of A.

Use diagonal matrices to compute matrix polynomials.

If $A = PBP^{-1}$, **then**

k **times**

$$A^k = PBP^{-1}PBP^{-1}\cdots PBP^{-1}PBP^{-1} = PB^kP^{-1}.$$

The polynomial in A **is**

$$\begin{aligned}\varphi(A) &= a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nE \\ &= a_0PB^nP^{-1} + a_1PB^{n-1}P^{-1} + \cdots \\ &\quad + a_{n-1}PBP^{-1} + a_nPEP^{-1} \\ &= P(a_0B^n + a_1B^{n-1} + \cdots + a_{n-1}B + a_nE)P^{-1} \\ &= P\varphi(B)P^{-1}.\end{aligned}$$

Particularly, if $P^{-1}AP = \Lambda$ **is diagonal, then**

$$A^k = P\Lambda^k P^{-1}, \quad \varphi(A) = P\varphi(\Lambda)P^{-1}.$$

Clearly, $\Lambda^k = \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix},$

$$\varphi(\Lambda) = \begin{pmatrix} \varphi(\lambda_1) & & & \\ & \varphi(\lambda_1) & & \\ & & \ddots & \\ & & & \varphi(\lambda_1) \end{pmatrix},$$

In this way, the polynomial $\varphi(A)$ **can be readily computed.**

Theorem. If $f(\lambda)$ is the characteristic polynomial of A, then $f(A)=0$.

Proof. We prove the conclusion under the condition that A is similar to a diagonal matrix.

In this case, there exists an invertible matrix P such that

$$P^{-1}AP=\Lambda=diag(\lambda_1,\cdots,\lambda_n),$$

where λ_i are the eigenvalues of A, and thus $f(\lambda_i)=0$.

From $A=P\Lambda P^{-1}$, it follows that

$$f(A)=Pf(\Lambda)P^{-1}=P\begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix}P^{-1}$$

$$=POP^{-1}=O.$$

3. Diagonalization of square matrices

A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = \Lambda$ is diagonal.

Theorem 2. **An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors.**

Proof. **If there is an invertible matrix P such that $P^{-1}AP = \Lambda$ is diagonal, let $P = (p_1, p_2, \cdots, p_n)$.**

From $P^{-1}AP = \Lambda$, it follows that $AP = P\Lambda$.

namely $A(p_1, p_2, \cdots, p_n) = (p_1, p_2, \cdots, p_n)\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$$= (\lambda_1 p_1, \lambda_2 p_2, \cdots, \lambda_n p_n).$$

Hence $A(p_1, p_2, \cdots, p_n) = (Ap_1, Ap_2, \cdots, Ap_n)$

$$= (\lambda_1 p_1, \lambda_2 p_2, \cdots, \lambda_n p_n).$$

Thus $Ap_i = \lambda_i p_i \quad (i = 1, 2, \cdots, n).$

Consequent ly, λ_i is eigenvalue of A, and the column vectors p_i of P is an eigenvecto r of A correspond ing to λ_i. Since P is invertible , $p_1, p_2, \cdots, p_n$ are linearly independen t.

Coversely, if A has n eigenvectors, then form the matrix P whose columns are n eigenvectors. Clearly, $AP = P\Lambda$, where Λ is a diagonal matrix with diagonal entries as eigenvalues.

Corollary.

If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

Remark.

If the characteristic polynomial of A has repeated roots， then it is possible that the matrix has less than n eigenvectors. As a result, the matrix is not necessarily diagonalizable.

Example. **Check whether the following matrices are diagonalizable or not.**

$$(1)\ A=\begin{pmatrix}1 & -2 & 2\\ -2 & -2 & 4\\ 2 & 4 & -2\end{pmatrix}\qquad (2)\ A=\begin{pmatrix}-2 & 1 & -2\\ -5 & 3 & -3\\ 1 & 0 & 2\end{pmatrix}$$

Solution.

(1) From $|A-\lambda E|=\begin{vmatrix}1-\lambda & -2 & 2\\ -2 & -2-\lambda & 4\\ 2 & 4 & -2-\lambda\end{vmatrix}$

$$=-(\lambda-2)^2(\lambda+7)=0$$

we have the eigenvalues $\lambda_1=\lambda_2=2,\ \lambda_3=-7.$

Substituting $\lambda = \lambda_1 = \lambda_2 = 2$ **into** $(A - \lambda E) = 0$ **yields**

$$\begin{cases} -x_1 - 2x_2 + 2x_3 = 0 \\ -2x_1 - 4x_2 + 4x_3 = 0 \\ 2x_1 + 4x_2 - 4x_3 = 0 \end{cases}$$

A system of fundamental solutions is

$$\alpha_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Similarly,

For $\lambda_3 = -7$, **solve the system** $(A - \lambda_3 E)x = 0$

to get a system of fundamental solutions $\alpha_3 = (1,2,2)^T$

Since $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} \neq 0$, $\alpha_1, \alpha_2, \alpha_3$ **are linearly independent.**

A **has 3 linearly independent eigenvectors, and thus** A **is diagonalizable.**

(2) $A = \begin{pmatrix} -2 & 1 & -2 \\ -5 & 3 & -3 \\ 1 & 0 & 2 \end{pmatrix}$

$$|A - \lambda E| = \begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = -(\lambda+1)^3$$

Hence the eigenvalues of A are $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

Substitute $\lambda = -1$ into $(A - \lambda E)x = 0$.

Solve it to obtain a system of fundamental solutions

$\xi = (1,1,-1)^T$. Hence A is not diagonalizable.

Example 2.

Let $A=\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$. **Check whether** A **is diagonalizable.**

If it is diagonalizable, find an invertible matrix P **such that** $P^{-1}AP$ **is a diagonal matrix .**

Solution.

$$|A-\lambda E| = \begin{vmatrix} 4-\lambda & 6 & 0 \\ -3 & -5-\lambda & 0 \\ -3 & -6 & 1-\lambda \end{vmatrix} = -(\lambda-1)^2(\lambda+2)$$

Thus all the eigenvalues of A **are** $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = -2.$

Substituting $\lambda_1 = \lambda_2 = 1$ **into** $(A - \lambda E)x = 0$ **leads to**

$$\begin{cases} 3x_1 + 6x_2 = 0 \\ -3x_1 - 6x_2 = 0 \\ -3x_1 - 6x_2 = 0 \end{cases}$$

whence we obtain a system of fundamental solutions

$$\xi_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Substitute $\lambda_3 = -2$ **into** $\left(A - \lambda E\right)x = 0$ **to obtain a system of fundamental solutions**

$$\xi_3 = \left(-1,1,1\right)^T.$$

Since ξ_1, ξ_2, ξ_3 **are linearly independent,** A **is diagonalizable.**

By letting $P = \left(\xi_1, \xi_2, \xi_3\right) = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$

we have $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$

Remark.

By letting $P=(\xi_3,\xi_1,\xi_2)=\begin{pmatrix}-1 & -2 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$,

we have $P^{-1}AP=\begin{pmatrix}-2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

In other words, the positions of the columns in the matrix correspond to those of the eigenvalues.