第10章 二阶电路的时域分析

本章重点

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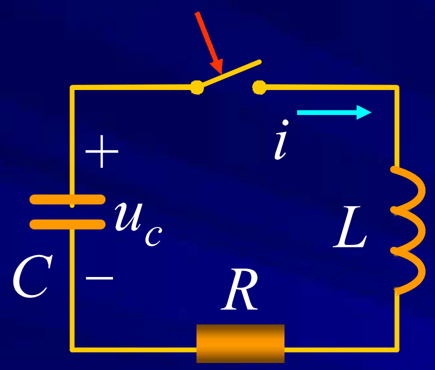
● 重点

1. 二阶电路的零输入响应、零状态响应和全响应的概念及求解;

2. 二阶电路的阶跃响应概念及求解。

10.1 二阶电路的零输入响应

1. 二阶电路的零输入响应



已知： $u_C(0_+)=U_0 \quad i(0_+)=0$

电路方程： $Ri + u_L - u_C = 0$

$$i = -C\frac{\mathrm{d}u_C}{\mathrm{d}t} \qquad u_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

以电容电压为变量： $LC\frac{\mathrm{d}^2u_C}{\mathrm{d}t} + RC\frac{\mathrm{d}u_C}{\mathrm{d}t} + u_C = 0$

以电感电流为变量： $LC\frac{\mathrm{d}^2i}{\mathrm{d}t} + RC\frac{\mathrm{d}i}{\mathrm{d}t} + i = 0$

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以电容电压为变量时的初始条件:

$$u_C(0_+)=U_0 \quad i(0_+)=0 \rightarrow \left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0_+}=0$$

以电感电流为变量时的初始条件:

$$i(0_+)=0 \quad u_C(0_+)=U_0 \rightarrow$$

$$u_C(0_+)=u_L(0_+)=L\left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{t=0_+}=U_0 \rightarrow \left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{t=0_+}=\frac{U_0}{L}$$

电路方程: $LC\frac{\mathrm{d}^2u_C}{\mathrm{d}t}+RC\frac{\mathrm{d}u_C}{\mathrm{d}t}+u_C=0$

特征方程: $LCP^2+RCP+1=0$

特征根: $P=\frac{-R\pm\sqrt{R^2-4L/C}}{2L}=-\frac{R}{2L}\pm\sqrt{(\frac{R}{2L})^2-\frac{1}{LC}}$

2. 零输入响应的三种情况

$R>2\sqrt{\frac{L}{C}}$ 二个不等负实根 过阻尼

$R=2\sqrt{\frac{L}{C}}$ 二个相等负实根 临界阻尼

$R<2\sqrt{\frac{L}{C}}$ 二个共轭复根 欠阻尼

(1) $R > 2\sqrt{\frac{L}{C}}$

$$u_C = A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

$$u_C(0_+) = U_0 \rightarrow A_1 + A_2 = U_0$$

$$\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{(0_+)} \rightarrow P_1 A_1 + P_2 A_2 = 0$$

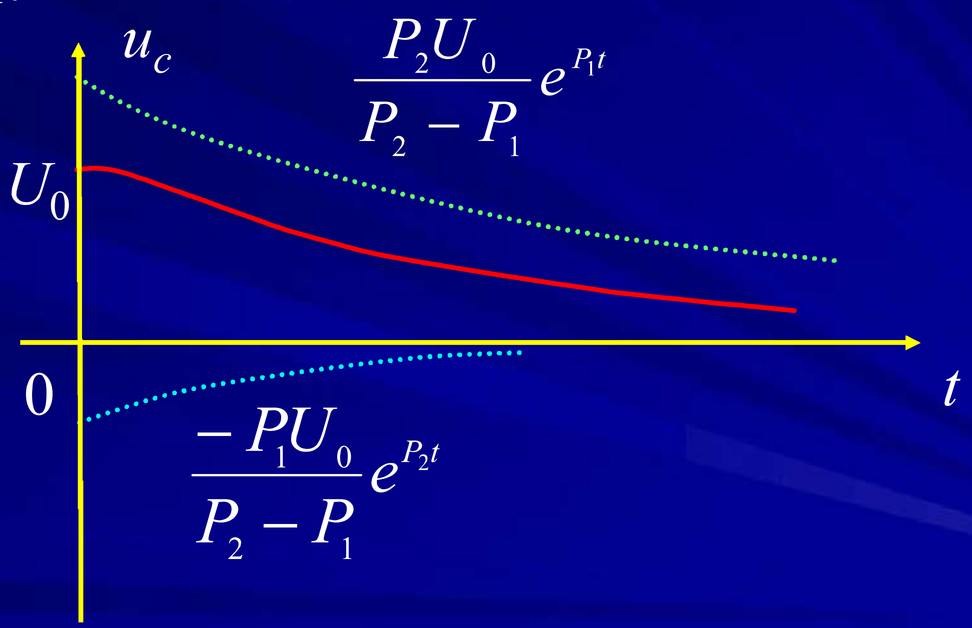
$$\begin{cases} A_1 = \dfrac{P_2}{P_2 - P_1} U_0 \\ A_2 = \dfrac{-P_1}{P_2 - P_1} U_0 \end{cases}$$

$$u_C = \frac{U_0}{P_2 - P_1}\left(P_2 e^{P_1 t} - P_1 e^{P_2 t}\right)$$

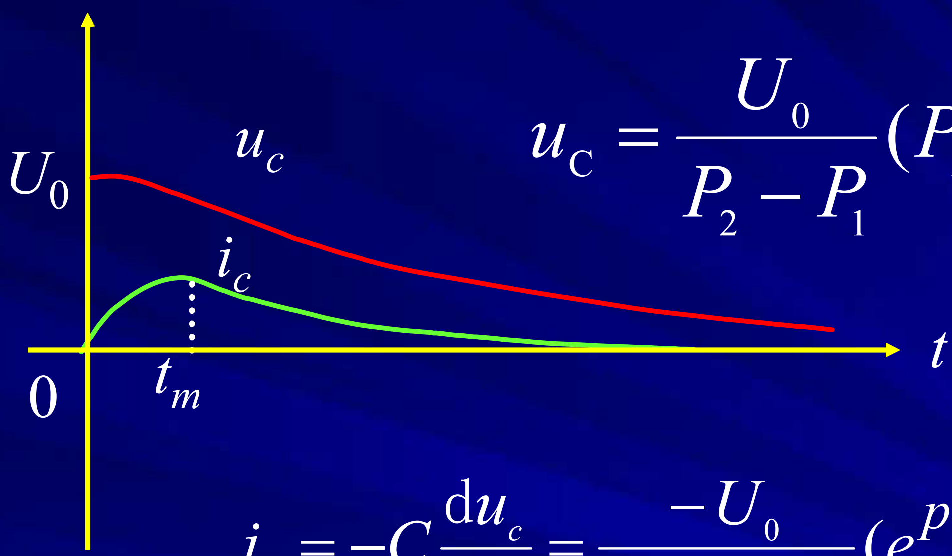
①电容电压

$$u_C = \frac{U_0}{P_2 - P_1}(P_2 e^{P_1 t} - P_1 e^{P_2 t})$$

设 $|P_2|>|P_1|$



②电容和电感电流



$$u_C = \frac{U_0}{P_2 - P_1}(P_2 e^{P_1 t} - P_1 e^{P_2 t})$$

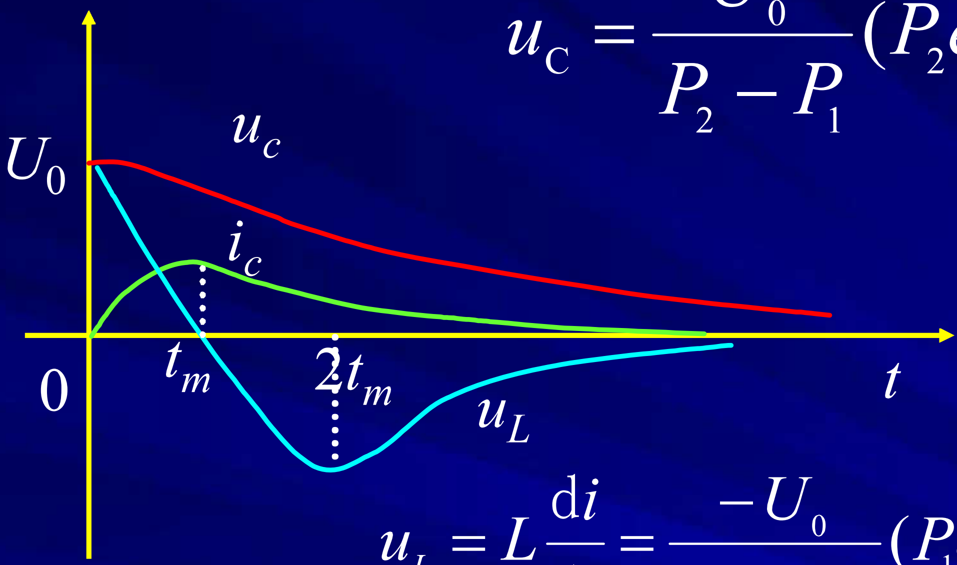
$$i_C = -C\frac{du_c}{dt} = \frac{-U_0}{L(P_2 - P_1)}(e^{p_1 t} - e^{p_2 t})$$

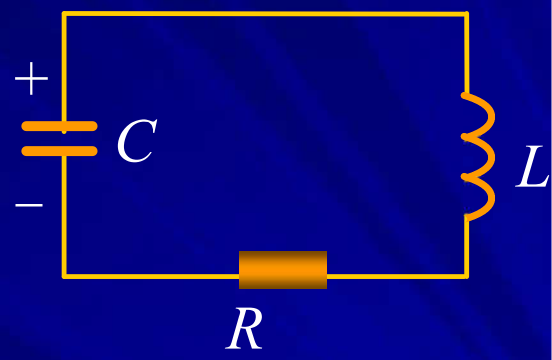
$t=0_+$　$i_c=0$　, $t=\infty$　$i_c=0$

$i_c>0$　$t = t_m$ 时 i_c 最大

③电感电压

$$u_C = \frac{U_0}{P_2 - P_1}(P_2 e^{P_1 t} - P_1 e^{P_2 t})$$





$$u_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{-U_0}{(P_2 - P_1)}(P_1 e^{p_1 t} - P_2 e^{p_2 t})$$

$t = 0,\ u_L = U_0 \quad t = \infty,\ u_L = 0$

$0 < t < t_m$，i 增加，$u_L > 0$， $t > t_m$ i 减小，$u_L < 0$

$t = 2t_m$ 时 $|u_L|$ 最大

$$u_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{-U_0}{(P_2 - P_1)}(P_1 e^{p_1 t} - P_2 e^{p_2 t})$$

$i_C=i$ 为极值时，即 $u_L=0$ 时的 t_m 计算如下：

$$(P_1 e^{p_1 t} - P_2 e^{p_2 t}) = 0 \qquad \frac{P_2}{P_1} = \frac{e^{P_1 t_m}}{e^{P_2 t_m}}$$

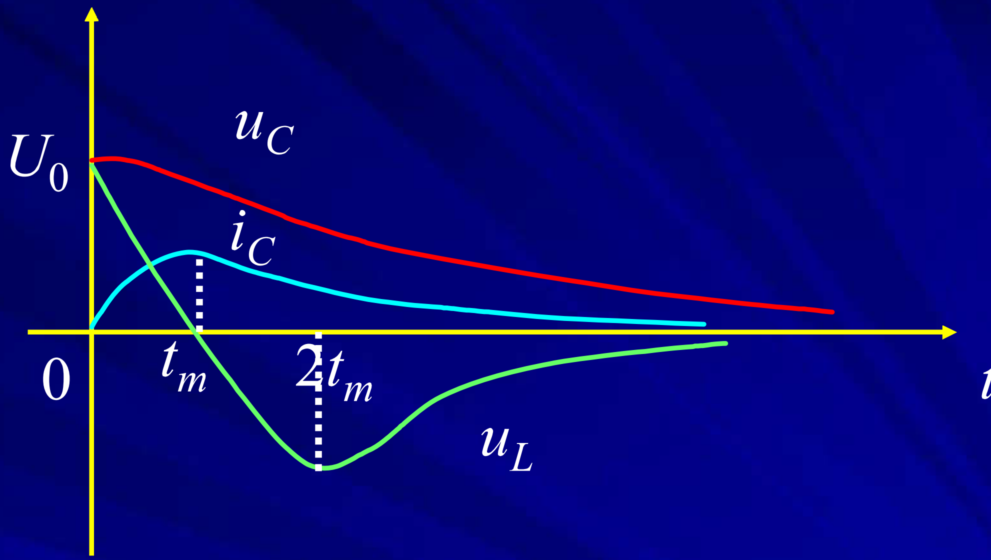
$$t_m = \frac{\ell n \frac{p_2}{p_1}}{p_1 - p_2}$$

由 $\mathrm{d}u_L/\mathrm{d}t$ 可确定 u_L 为极小时的 t .

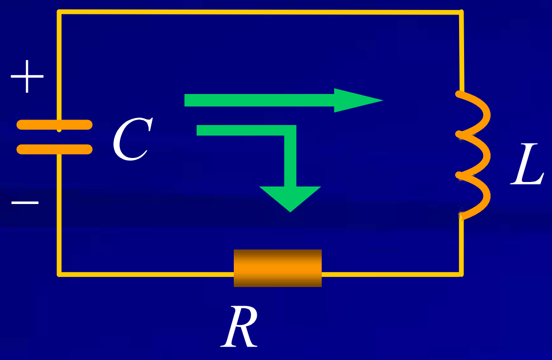
$$(P_1^2 e^{p_1 t} - P_2^2 e^{p_2 t}) = 0 \qquad t = \frac{2\ell n \frac{p_2}{p_1}}{p_1 - p_2}$$

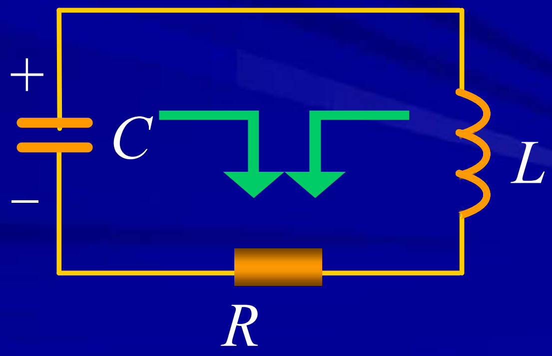
$$\rightarrow \quad t = 2t_m$$

④能量转换关系



$0 < t < t_m$　u_C减小，i增加。$t > t_m$　u_C减小，i减小.





(2) $R < 2\sqrt{\frac{L}{C}}$

$$P_{1,2} = -\frac{R}{2L} \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}$$

共轭复根

令：$\delta = \frac{R}{2L}$（衰减系数），$\omega_0 = \sqrt{\frac{1}{LC}}$（谐振角频率）

$\omega = \sqrt{\omega_0^2 - \delta^2}$（固有振荡角频率）　　$P = -\delta \pm j\omega$

u_c 的解答形式:

$$u_C = A_1 e^{p_1 t} + A_2 e^{p_2 t} = e^{-\delta(t)}(A_1 e^{j\omega t} + A_2 e^{-j\omega t})$$

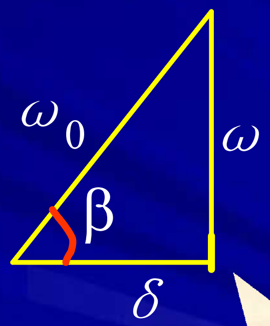
经常写为:

$$u_C = Ae^{-\delta t}\sin(\omega t + \beta)$$

$$u_C = \frac{\omega_0}{\omega} U_0 e^{-\delta t} \sin(\omega t + \beta)$$

由初始条件
$$\begin{cases} u_C(0^+) = U_0 \rightarrow A\sin\beta = U_0 \\ \dfrac{du_C}{dt}(0^+) = 0 \rightarrow A(-\delta)\sin\beta + A\omega\cos\beta = 0 \end{cases}$$

$$A = \frac{U_0}{\sin\beta}\quad , \quad \beta = arctg\frac{\omega}{\delta}$$



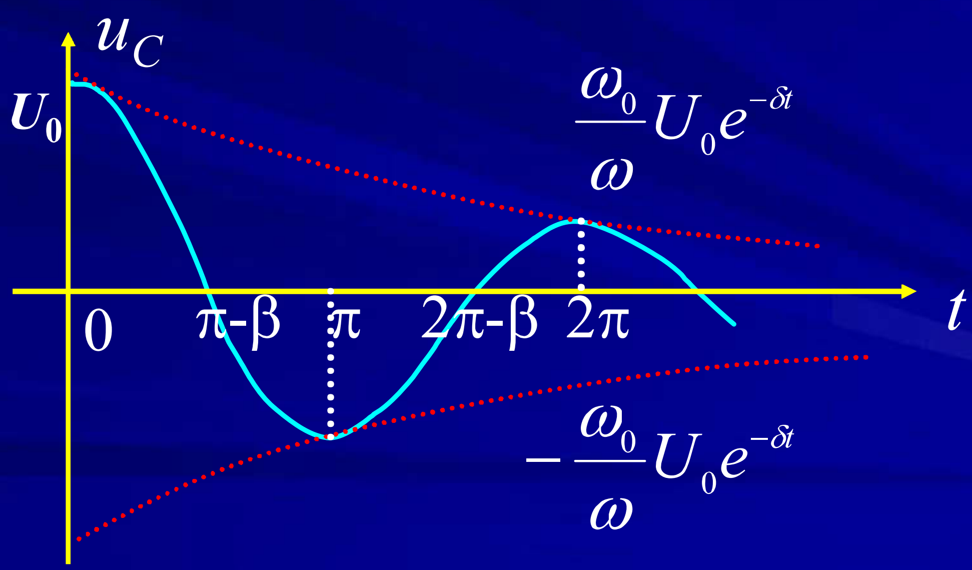
$$\sin\beta = \frac{\omega}{\omega_0} \qquad A = \frac{\omega_0}{\omega} U_0$$

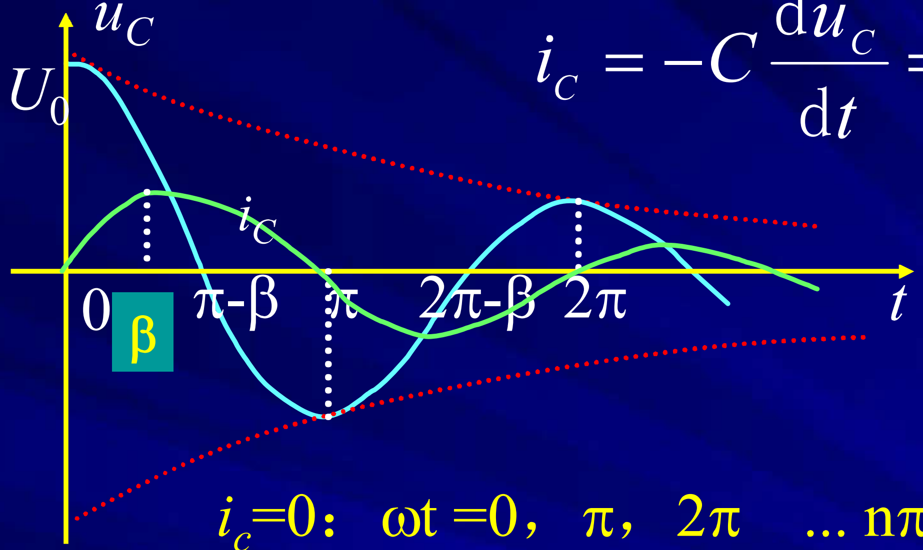
ω，ω_0，δ 的关系

$$u_C = \frac{\omega_0}{\omega} U_0 e^{-\delta t} \sin(\omega t + \beta)$$

u_C是振幅以$\pm \frac{\omega_0}{\omega} U_0$为包线依指数衰减的正弦函数。

$t=0$ 时 $u_c=U_0$　　　$u_C=0$：$\omega t = \pi-\beta，2\pi-\beta \; ... \; n\pi-\beta$





$$i_C = -C\frac{\mathrm{d}u_C}{\mathrm{d}t} = \frac{U_0}{\omega L}e^{-\delta t}\sin\omega t$$

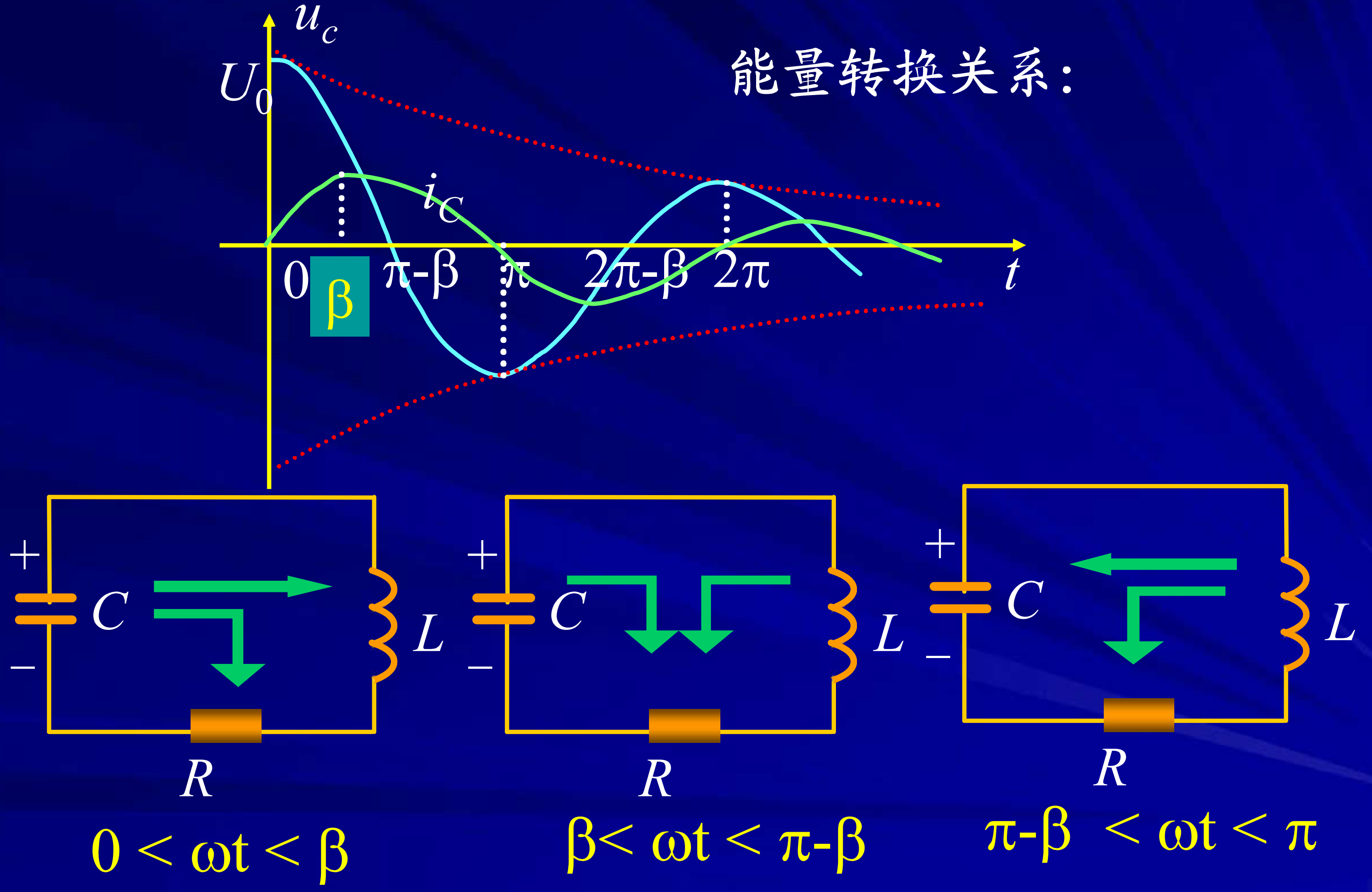
i_c=0：ωt =0，π，2π　... nπ，为 u_c极值点，

i_c 的极值点为 u_L 零点。

$$u_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{\omega_0}{\omega}U_0 e^{-\delta t}\sin(\omega t - \beta)$$

u_L=0：ωt = β，π+β，2π+β　... nπ+β

能量转换关系：

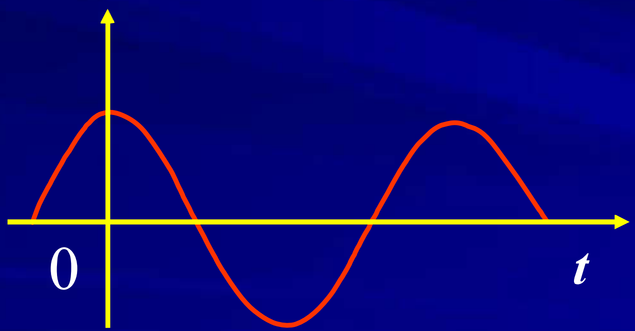


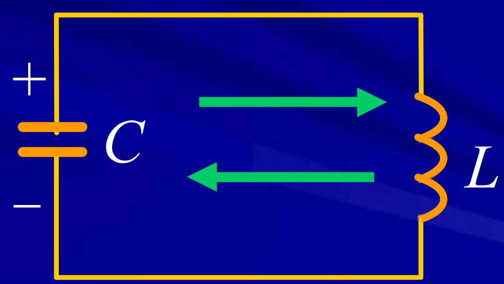
特例：$R=0$ 时　$\delta=0$，$\omega=\omega_0=\dfrac{1}{\sqrt{LC}}$，$\beta=\dfrac{\pi}{2}$

$$u_C=U_0\sin(\omega t+90^0)=u_L$$

$$i=\frac{U_0}{\omega L}\sin\omega t$$

→ 等幅振荡





(3) $R = 2\sqrt{\frac{L}{C}}$

$$P_1 = P_2 = -\frac{R}{2L} = -\delta$$

相等负实根

$$u_C = A_1 e^{-\delta t} + A_2 t e^{-\delta t}$$

由初始条件
$$\begin{cases} u_c(0^+) = U_0 \rightarrow A_1 = U_0 \\ \dfrac{\mathrm{d}u_c}{\mathrm{d}t}(0^+) = 0 \rightarrow A_1(-\delta) + A_2 = 0 \end{cases}$$

$$\begin{cases} A_1 = U_0 \\ A_2 = U_0 \delta \end{cases}$$

$$u_C = A_1 e^{-\delta t} + A_2 t e^{-\delta t}$$

$$A_1 = U_0 \quad A_2 = U_0 \delta$$

$$\left.\begin{aligned} u_C &= U_0 e^{-\delta t}(1+\delta t) \\ i_C &= -C\frac{\mathrm{d}u_C}{\mathrm{d}t} = \frac{U_0}{L} t e^{-\delta t} \\ u_L &= L\frac{\mathrm{d}i}{\mathrm{d}t} = U_0 e^{-\delta t}(1-\delta t) \end{aligned}\right\} \text{非振荡放电}$$

小结

$R > 2\sqrt{\dfrac{L}{C}}$ 过阻尼，　非振荡放电

$$u_C = A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

$R = 2\sqrt{\dfrac{L}{C}}$ 临界阻尼，　非振荡放电

$$u_C = A_1 e^{-\delta t} + A_2 t e^{-\delta t}$$

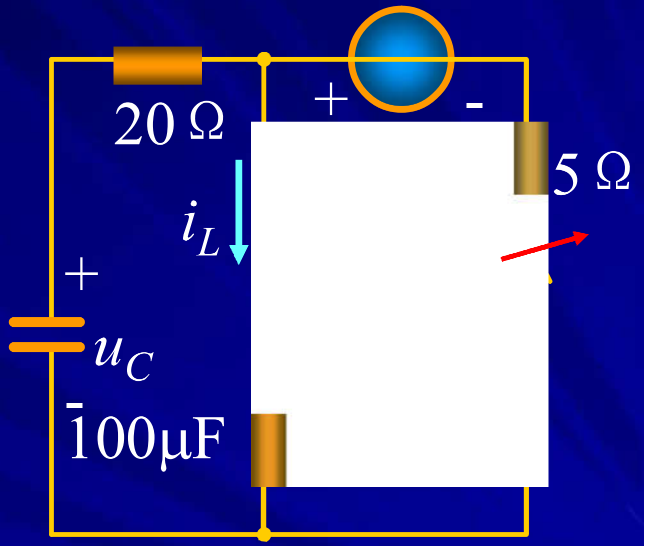
$R < 2\sqrt{\dfrac{L}{C}}$ 欠阻尼，　振荡放电

$$u_C = A e^{-\delta t} \sin(\omega t + \beta)$$

可推广应用于一般二阶电路

由初始条件 $\begin{cases} u_C(0_+) \\ \dfrac{\mathrm{d}u_C}{\mathrm{d}t}(0_+) \end{cases}$ 定常数

例1　电路如图，t=0 时打开开关。求 u_C并画出其变化曲线。



解　（1） $u_C(0_-)=25\text{V}$

$i_L(0_-)=5\text{A}$

（2）开关打开为*RLC*串联电路，方程为：

$$LC\frac{\mathrm{d}^2u_C}{\mathrm{d}t}+RC\frac{\mathrm{d}u_C}{\mathrm{d}t}+u_C=0$$

特征方程为： $50P^2+2500P+10^6=0$

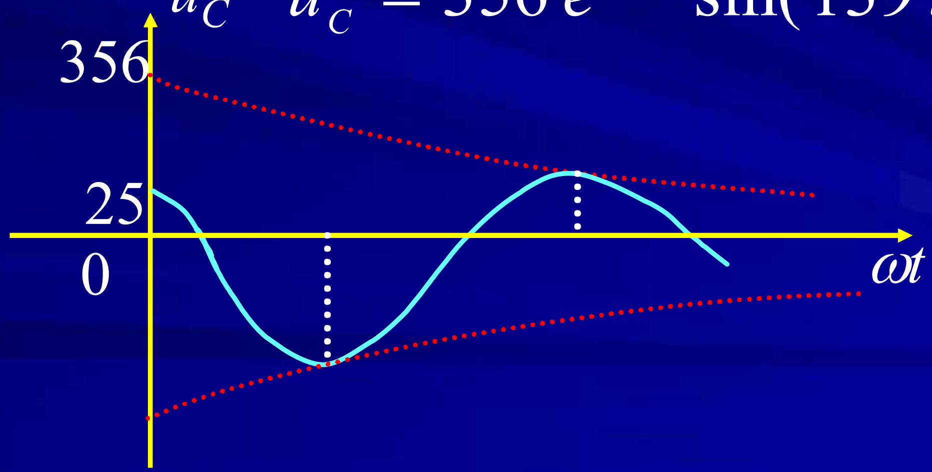
$$P=-25\pm j139 \qquad u_C=Ae^{-25t}\sin(139t+\beta)$$

$$u_C = Ae^{-25t}\sin(139t+\beta)$$

(3) $\begin{cases} u_C(0_+) = 25 \\ C\dfrac{\mathrm{d}u_C}{\mathrm{d}t}\Big|_{0_+} = -5 \end{cases}$ → $\begin{cases} A\sin\beta = 25 \\ A(139\cos\beta - 25\sin\beta) = \dfrac{-5}{10^{-4}} \end{cases}$

$$A = 356,\quad \beta = 176^0$$

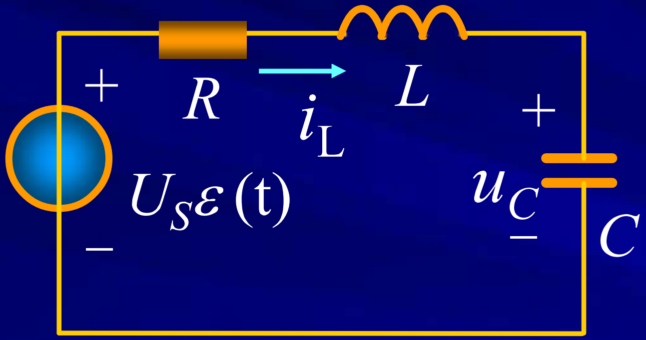
$$u_C = 356e^{-25t}\sin(139t+176^0)\,\mathrm{V}$$



10.2 二阶电路的零状态响应和全响应

1. 二阶电路的零状态响应

例　$u_C(0_-)=0$， $i_L(0_-)=0$



微分方程为:

$$LC\frac{d^2u_C}{dt}+RC\frac{du_C}{dt}+u_C=U_s$$

特征方程为:

$$LCP^2+RCP+1=0$$

$$u_C=u'_C+u''_C$$

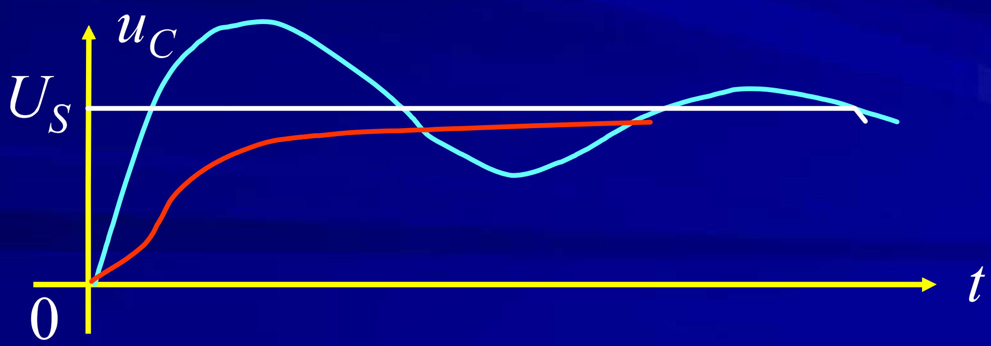
特解　通解

特解: $u'_C=U_s$

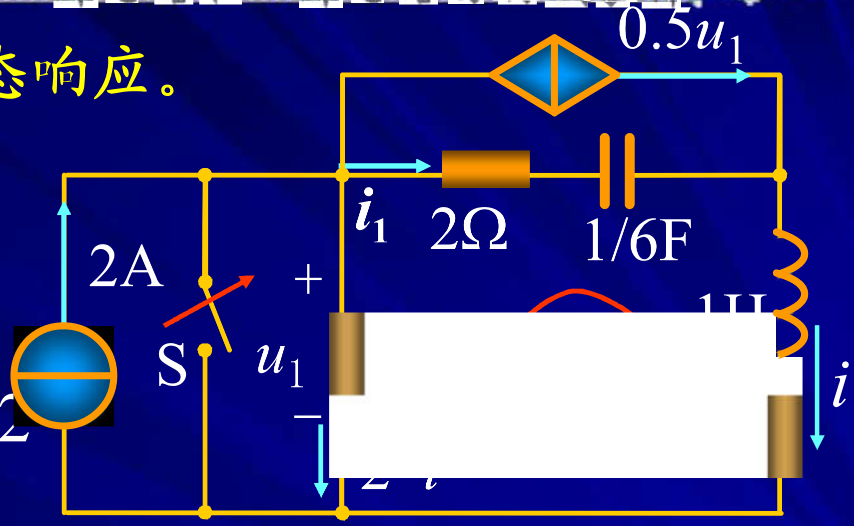
u_C解答形式为:

$$\begin{cases} u_C = U_s + A_1 e^{p_1 t} + A_2 e^{p_2 t} & (p_1 \neq p_2) \\ u_C = U_s + A_1 e^{-\delta t} + A_2 t e^{-\delta t} & (P_1 = P_2 = -\delta) \\ u_C = U_s + A e^{-\delta t} \sin(\omega t + \beta) & (P_{1、2} = -\delta \pm j\omega) \end{cases}$$

由初值 $u_C(0_+)$, $\dfrac{\mathrm{d}u(0_+)}{\mathrm{d}t}$ 确定二个常数



例　求电流 i 的零状态响应。



解　首先写微分方程

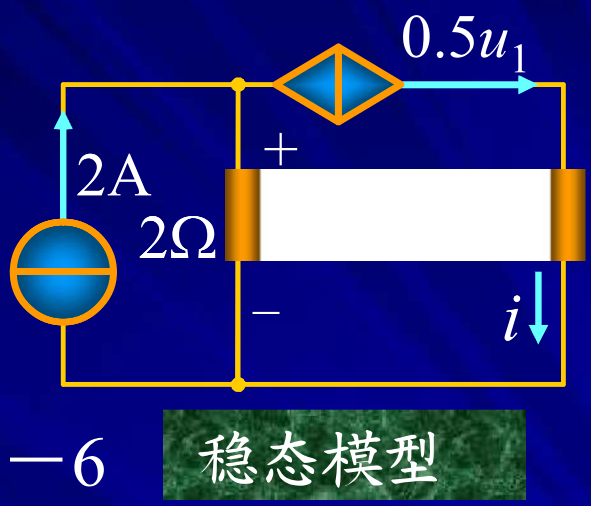
$$i_1 = i - 0.5\,u_1$$
$$= i - 0.5(2 - i)\times 2$$
$$= 2i - 2$$

由KVL：$$2(2-i) = 2i_1 + 6\int i_1 dt + \frac{\mathrm{d}i}{\mathrm{d}t} + 2i$$

整理得：$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + 8\frac{\mathrm{d}i}{\mathrm{d}t} + 12i = 12$$

二阶非齐次常微分方程

$$\frac{d^2 i}{dt^2} + 8\frac{di}{dt} + 12i = 12$$



解答形式为：　$i = i' + i''$

第二步求通解 i''

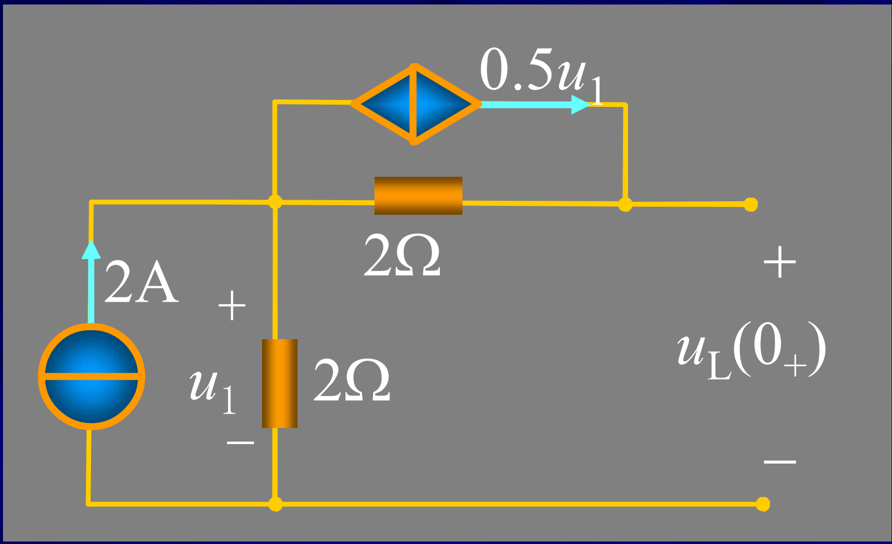
特征根为：$P_1 = -2$ ，$P_2 = -6$

$$i'' = A_1 e^{-2t} + A_2 e^{-6t}$$

第三步求特解 i'

由稳态模型有：$i' = 0.5\, u_1$　　$u_1 = 2(2 - 0.5u_1)$

→　$u_1 = 2$　　$i' = 1\text{A}$



第四步定常数

$$i = 1 + A_1 e^{-2t} + A_2 e^{-6t}$$

$$\begin{cases} i(0_+) = i(0_-) = 0 \\ L\dfrac{\mathrm{d}i}{\mathrm{d}t}(0_+) = u_L(0_+) \end{cases}$$

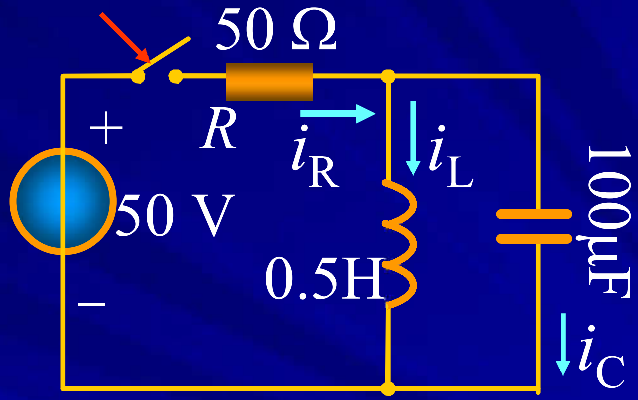
由0_+电路模型： $u_L(0_+) = 0.5u_1 \times 2 + u_1 = 2u_1 = 8\text{V}$

$$\begin{cases} 0 = 1 + A_1 + A_2 \\ 8 = -2A_1 - 6A_2 \end{cases} \qquad \begin{cases} A_1 = 0.5 \\ A_2 = -1.5 \end{cases}$$

$$\therefore \quad i = 1 + 0.5e^{-2t} - 1.5e^{-6t} \quad \text{A}$$

2. 二阶电路的全响应

例　已知：$i_L(0_-)=2A$　$u_C(0_-)=0$　求：i_L，i_R



解　(1) 列微分方程

应用结点法：

$$\frac{L\frac{di_L}{dt}-50}{R}+i_L+LC\frac{d^2i_L}{dt^2}=0$$

$$RLC\frac{d^2i_L}{dt^2}+L\frac{di_L}{dt}+Ri_L=50$$

(2) 求特解

$$i'_L=1A$$

$$RLC\frac{d^2 i_L}{dt^2} + L\frac{di}{dt} + Ri_L = 50$$

(3) 求通解　特征方程为：$P^2 + 200P + 20000 = 0$

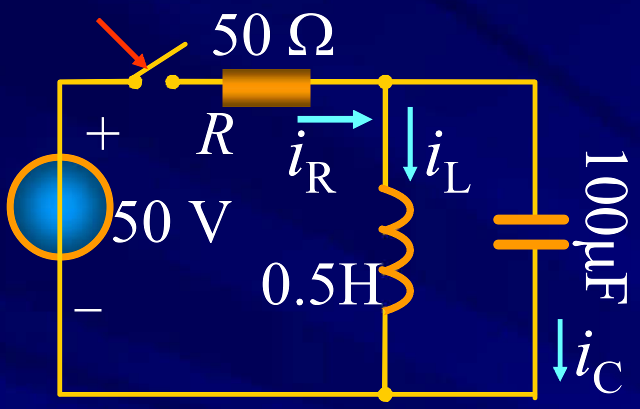
特征根为：$P= -100 \pm j100$

$$\rightarrow \quad i = 1 + Ae^{-100t}\sin(100t + \varphi)$$

(4) 定常数

$$\begin{cases} 1 + A\sin\varphi = 2 & \leftarrow i_L(0_+) \\ 100A\cos\varphi - 100A\sin\varphi = 0 & \leftarrow u_L(0_+) \end{cases}$$

$$\begin{cases} \varphi = 45^\circ \\ A = \sqrt{2} \end{cases} \quad \rightarrow \quad i_L = 1 + \sqrt{2}e^{-100t}\sin(100t + 45^\circ)$$

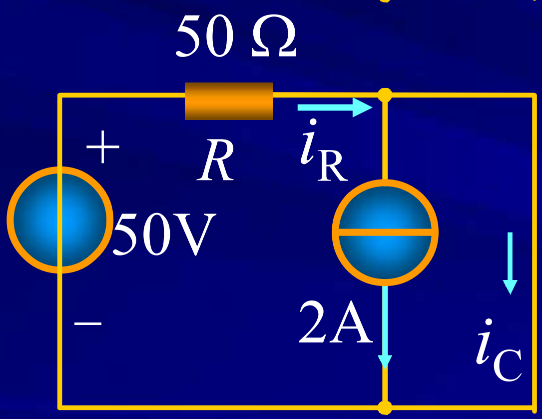


(5) 求i_R

$$i_R = i_L + i_C = i_L + LC\frac{d^2 i_L}{dt^2}$$

或设解答形式为:

$$i_R = 1 + Ae^{-100t}\sin(100t+\varphi)$$



定常数

$$\begin{cases} i_R(0_+) = 1 \quad i_C(0_+) = -1 \\ \dfrac{di_R}{dt}(0_+) = ? \quad i_R = \dfrac{50-u_C}{R} \end{cases}$$

$$\frac{di_R}{dt}(0_+) = -\frac{1}{R}\frac{du_C}{dt}(0_+) = -\frac{1}{RC}i_C(0_+) = 200$$

$$i_R = 1 + Ae^{-100t}\sin(100t + \varphi)$$

$$\begin{cases} 1 + A\sin\varphi = 1 \\ 100A\cos\varphi - 100A\sin\varphi = 200 \end{cases} \qquad \begin{cases} \varphi = 0 \\ A = 2 \end{cases}$$

小结

1. 二阶电路含二个独立储能元件，是用二阶常微分方程所描述的电路。
2. 二阶电路的性质取决于特征根，特征根取决于电路结构和参数，与激励和初值无关。

$$p = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

$\delta > \omega_0$ 过阻尼， 非振荡放电 $u_C = A_1 e^{p_1 t} + A_2 e^{p_2 t}$

$\delta = \omega_0$ 临界阻尼， 非振荡放电 $u_C = A_1 e^{-\delta t} + A_2 t e^{-\delta t}$

$\delta < \omega_0$ 欠阻尼， 振荡放电 $u_C = A e^{-\delta t} \sin(\omega t + \beta)$

3. 求二阶电路全响应的步骤

(a) 列写 $t>0_+$ 电路的微分方程

(b) 求通解

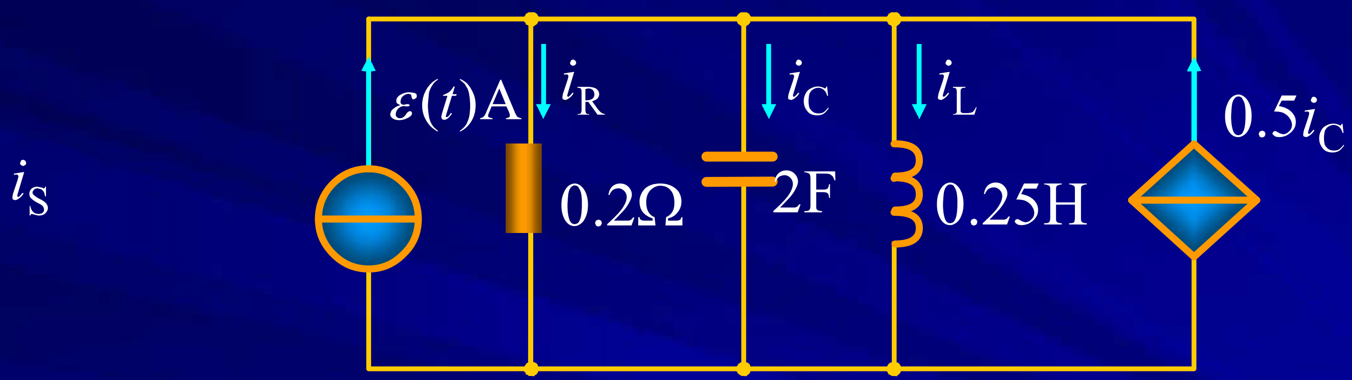
(c) 求特解

(d) 全响应=强制分量+自由分量

(e)由初值 $\left.\begin{array}{l} f(0_+) \\ \dfrac{df}{dt}(0_+) \end{array}\right\rangle$ 定常数

10.3. 二阶电路的阶跃响应

例 已知图示电路中 $u_C(0_-)=0$，$i_L(0_-)=0$，求单位阶跃响应 $i_L(t)$



解 对电路应用KCL列结点电流方程有

$$i_R + i_C + i_L - 0.5 i_C = i_S$$

$$i_R + 0.5 i_C + i_L = \varepsilon(t)$$

$$i_R = \frac{u_R}{R} = \frac{L}{R}\frac{\mathrm{d}i_L}{\mathrm{d}t} \qquad i_C = C\frac{\mathrm{d}u_C}{\mathrm{d}t} = LC\frac{\mathrm{d}^2 i_L}{\mathrm{d}t^2}$$

代入已知参数并整理得：
$$\frac{\mathrm{d}i_L^2}{\mathrm{d}t^2} + 5\frac{\mathrm{d}i_L}{\mathrm{d}t} + 4i_L = 4\varepsilon(t)$$

这是一个关于的二阶线性非齐次方程，其解为

$$i_L = i' + i''$$

特解 $i' = 1$ 通解 $i'' = A_1 \mathrm{e}^{p_1 t} + A_2 \mathrm{e}^{p_2 t}$

特征方程 $p^2 + 5p + 4 = 0$

解得特征根 $p_1 = -1 \qquad p_2 = -4$

$$i_L = 1 + A_1 \mathrm{e}^{-t} + A_2 \mathrm{e}^{-4t}$$

代初始条件 $i_L(0_+) = i_L(0_-) = 0$

$$u_C(0_+) = u_C(0_-) = 0$$

$$\rightarrow \begin{cases} 1 + A_1 + A_2 = 0 \\ -A_1 - 4A_2 = 0 \end{cases} \rightarrow \begin{cases} A_1 = -\dfrac{4}{3} \\ A_2 = \dfrac{1}{3} \end{cases}$$

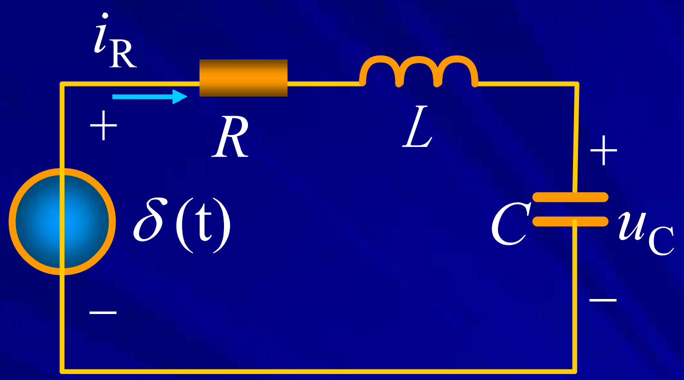
阶跃响应 $i_L(t) = s(t) = \left(1 - \dfrac{4}{3}\mathrm{e}^{-t} + \dfrac{1}{3}\mathrm{e}^{-4t}\right)\varepsilon(t)\mathrm{A}$

电路的动态过程是过阻尼性质的。

4. 二阶电路的冲激响应

例 求单位冲激电压激励下的*RLC*电路的零状态响应。

解　KVL方程为



$$LC\frac{d^2u_C}{dt^2}+RC\frac{du_C}{dt}+u_C=\delta(t)$$

$$\int_{0_-}^{0_+}LC\frac{d^2u_C}{dt^2}dt+\int_{0_-}^{0_+}RC\frac{du_C}{dt}dt+\int_{0_-}^{0_+}u_Cdt=\int_{0_-}^{0_+}\delta(t)dt$$

有限值　　有限值

t 在0_-至0_+间

$$\int_{0_-}^{0_+}LC\frac{d^2u_C}{dt^2}dt=1$$

$$\int_{0_-}^{0_+} LC\frac{\mathrm{d}^2u_C}{\mathrm{d}t^2}\mathrm{d}t = 1 \rightarrow LC\frac{\mathrm{d}u_C}{\mathrm{d}t}(0_+) - LC\frac{\mathrm{d}u_C}{\mathrm{d}t}(0_-) = 1$$

$$\rightarrow i_L(0_+) = i_C(0_+) = \frac{1}{L} \qquad u_C(0_+) = u_C(0_-) = 0$$

$t>0_+$为零输入响应

$$LC\frac{\mathrm{d}^2u_C}{\mathrm{d}t^2} + RC\frac{\mathrm{d}u_C}{\mathrm{d}t} + u_C = 0$$

$$R > 2\sqrt{\frac{L}{C}} \rightarrow u_C = A_1e^{p_1t} + A_2e^{p_2t}$$

$$\begin{cases} A_1 + A_2 = 0 \\ A_1P_1 + A_2P_2 = \dfrac{1}{LC} \end{cases} \qquad A_2 = -A_1 = \frac{\dfrac{1}{LC}}{P_2 - P_1}$$

$$u_C = \frac{-1}{LC(P_2 - P_1)}(e^{p_1 t} - e^{p_2 t})\varepsilon(t)$$

$R < 2\sqrt{\frac{L}{C}}$ $\qquad (P_{1,2} = -\delta \pm \mathrm{j}\omega)$

$$u_C = Ae^{-\delta t}\sin(\omega t + \beta)$$

$$\rightarrow \quad u_C = \frac{1}{\omega LC}e^{-\delta t}\sin(\omega t)\varepsilon(t)$$