Space-time, relativity and quantum mechanics: In search of a deeper connection

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Abstract

It has been shown that the Lorentz transformations in special relativity can be derived in terms of the principle of relativity and certain properties of space and time such as homogeneity. In this paper, we argue that the free Schrödinger equation in quantum mechanics may also be regarded as a consequence of the homogeneity of space and time and the principle of relativity when assuming linearity of time evolution.

1 Introduction

It is a fundamental assumption in modern physics that space and time are homogeneous. The homogeneity of space and time is reflected in the spacetime translation invariance of natural laws, and it ensures that the same experiment performed at two different places or repeated at two different times gives the same result. It has been widely demonstrated that the Lorentz transformations in special relativity can be derived in terms of the principle of relativity and certain properties of space and time such as homogeneity and isotropy (see, e.g. Torretti 1983; Brown 2005). In this paper, we will investigate the implications of the homogeneity of space and time for quantum mechanics, and in particular, we will argue that the wave equation for free particles in the theory may be regarded as a consequence of spacetime translation invariance and relativistic invariance when assuming linearity of time evolution.

The plan of this paper is as follows. In Section 2, spacetime translation invariance is analyzed. It is well known that spacetime translation gives the definitions of momentum and energy in quantum mechanics, and the momentum operator P and energy opertaor H are defined as the generators of space translation and time translation, respectively. Here we argue that spacetime translation invariance entails that the state of a free particle with definite momentum and energy assumes the plane wave form $e^{i(px-Et)}$ when assuming the

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time evolution of the state is linear. In Section 3, we show that the relativistic invariance of the free state may further determine the relativistic energy-momentum relation. In Section 4, we obtain the Klein-Gorden equation for free particles in relativistic quantum mechanics based on these results. This equation reduces to the free particle Schrödinger equation in the nonrelativistic domain. Conclusions are given in the last section.

2 Spacetime translation invariance

There are in general two different pictures of translation: active transformation and passive transformation. The active transformation corresponds to displacing the studied system, and the passive transformation corresponds to moving the coordinate system. Physically, the equivalence of the active and passive pictures is due to the fact that moving the system one way is equivalent to moving the coordinate system the other way by an equal amount. In the following, we will mainly analyze spacetime translations in terms of active transformations.

A space translation operator can be defined as

$$T(a)\psi(x,t) = \psi(x-a,t). \tag{1}$$

It means translating rigidly the state of a system, $\psi(x,t)$, by an amount a in the positive x direction[1]. $T(a)$ can be further expressed as

$$T(a) = e^{-iaP}, \tag{2}$$

where P is called the generator of space translation[2]. By expanding $\psi(x-a,t)$ in order of a, we can further get

$$P = -i\frac{\partial}{\partial x}. \tag{3}$$

Similarly, a time translation operator can be defined as

$$U(t)\psi(x,0) = \psi(x,t). \tag{4}$$

Let the evolution equation of state be of the following form:

$$i\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t). \tag{5}$$

where H is a to-be-determined operator that depends on the properties of the system[3]. Then the time translation operator $U(t)$ can be expressed as $U(t) = e^{-itH}$, and H is the generator of time translation. In the following analysis, we assume H is independent of the evolved state, namely the evolution is linear[4].

[1]Note that this form of state is general, e.g. it also includes the state of continuous motion of a particle, for which the state may be $\psi(x,t) = \delta(x - x(t))$, where $x(t)$ is the trajectory of the particle.

[2]For convenience of later discussions we introduce the imaginary unit i in the expression. This does not influence the validity of the following analysis.

[3]Similarly we also introduce the imaginary unit i in the equation for convenience of later discussions.

[4]Note that the linearity of H is an important presupposition in our derivation of the free Schrödinger equation. It can be reasonably assumed that the linear evolution and nonlinear evolution both exist, and moreover, they satisfy spacetime translation invariance respectively

Let's now see the implications of spacetime translation invariance[5]. First, time translational invariance requires that H have no time dependence, namely $dH/dt = 0$. This can be demonstrated as follows (see also Shankar 1994). Suppose an isolated system is in state ψ_0 at time t_1 and evolves for an infinitesimal time δt. The state of the system at time $t_1 + \delta t$, to first order in δt, will be

$$\psi(x, t_1 + \delta t) = [I - i\delta t H(t_1)]\psi_0 \tag{6}$$

If the evolution is repeated at time t_2, beginning with the same initial state, the state at $t_2 + \delta t$ will be

$$\psi(x, t_2 + \delta t) = [I - i\delta t H(t_2)]\psi_0 \tag{7}$$

Time translational invariance requires the outcome state should be the same:

$$\psi(x, t_2 + \delta t) - \psi(x, t_1 + \delta t) = i\delta t[H(t_1) - H(t_2)]\psi_0 = 0 \tag{8}$$

Since the initial state ψ_0 is arbitrary, it follows that $H(t_1) = H(t_2)$. Moreover, since t_1 and t_2 are also arbitrary, it follows that H is time-independent, namely $dH/dt = 0$. It can be seen that this result relies on the linearity of time evolution. If H depends on the state, then obviously we cannot obtain $dH/dt = 0$ because the state is time-dependent, though we still have $H(t_1, \psi_0) = H(t_2, \psi_0)$, which means that the state-dependent H also satisfies time translational invariance.

Secondly, space translational invariance requires $[T(a), U(t)] = 0$, which further leads to $[P, H] = 0$. This can be demonstrated as follows (see also Shankar 1994). Suppose at $t = 0$ two observers A and B prepare identical isolated systems at $x = 0$ and $x = a$, respectively. Let $\psi(x, 0)$ be the state of the system prepared by A. Then $T(a)\psi(x, 0)$ is the state of the system prepared by B, which is obtained by translating (without distortion) the state $\psi(x, 0)$ by an amount a to the right. The two systems look identical to the observers who prepared them. After time t, the states evolve into $U(t)\psi(x, 0)$ and $U(t)T(a)\psi(x, 0)$. Since the time evolution of each identical system at different places should appear the same to the local observers, the above two systems, which differed only by a spatial translation at $t = 0$, should differ only by the same spatial translation at future times. Thus the state $U(t)T(a)\psi(x, 0)$ should be the translated version of A's system at time t, namely we have $U(t)T(a)\psi(x, 0) = T(a)U(t)\psi(x, 0)$. This relation holds true for any initial state $\psi(x, 0)$, and thus we have $[T(a), U(t)] = 0$, which says that space translation operator and time translation operator are commutative. Again, we note that the linearity of time evolution is an important presupposition of this result. If $U(t)$ depends on the state, then the space translational invariance will only lead to $U(t, T\psi)T(a)\psi(x, 0) = T(a)U(t, \psi)\psi(x, 0)$, from which we cannot obtain $[T(a), U(t)] = 0$.

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because they cannot counteract each other in general. Then our following analysis will show that the linear evolution part, if it exists, must assume the same form as the free Schrödinger equation in the nonrelativistic domain. Certainly, our derivation cannot exclude the existence of possible nonlinear evolution.

[5]The evolution law of an isolated system satisfies spacetime translation invariance due to the homogeneity of space and time. The homogeneity of space ensures that the same experiment performed at two different places gives the same result, and the homogeneity in time ensures that the same experiment repeated at two different times gives the same result.

When $dH/dt = 0$, the solutions of the evolution equation Eq.(5) assume the following form

$$\psi(x,t) = \varphi_E(x)e^{-iEt} \tag{9}$$

and superpositions thereof, where E is a constant, and $\varphi_E(x)$ is the eigenstate of H and satisfies the time-independent equation:

$$H\varphi_E(x) = E\varphi_E(x). \tag{10}$$

The commutative relation $[P, H] = 0$ further implies that P and H have common eigenstates. This means that $\varphi_E(x)$ is also the eigenstate of P. Since the eigenstate of $P \equiv -i\frac{\partial}{\partial x}$ is e^{ipx} (except a normalization factor), where p is an eigenvalue, the solutions of the evolution equation Eq.(5) for an isolated system will be $e^{i(px-Et)}$. In quantum mechanics, P and H, the generators of space translation and time translation, are also called momentum operator and energy operator, respectively. Correspondingly, $e^{i(px-Et)}$ is the eigenstate of both momentum and energy, and p and E are the corresponding momentum and energy eigenvalues, respectively. In other words, the state $e^{i(px-Et)}$ describes an isolated system (e.g. a free electron) with definite momentum p and energy E.

3 Relativistic invariance

The relation between momentum p and energy E can be determined by the relativistic invariance of the free state $e^{i(px-Et)}$, and it turns out to be $E^2 = p^2c^2 + m^2c^4$, where m is the rest mass of the system, and c is the speed of light[6]. In the nonrelativistic domain, the energy-momentum relation reduces to $E = p^2/2m$.

Now we will determine the relation between momentum p and energy E in the relativistic domain. Consider two inertial frames S_0 and S with coordinates x_0, t_0 and x, t. S_0 is moving with velocity v relative to S. Then x, t and x_0, t_0 satisfy the Lorentz transformations:

$$x_0 = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{11}$$

$$t_0 = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \tag{12}$$

As noted before, the Lorentz transformations can be derived in terms of certain properties of space and time such as homogeneity and isotropy and the principle of relativity. For the purpose of this paper, here we only illustrate the role of homogeneity of space and time in the derivation. That space-time is homogeneous requires that it has the same properties "everywhere and every time". More precisely, the transformation properties of a spatiotemporal

[6]Different from the derivation given here, most existing derivations of the energy-momentum relation are based on a somewhat complex analysis of an elastic collision process and the principle of conservation of momentum and energy. Moreover, they resort to either some Newtonian limit (e.g. $p = mv$) or some less fundamental relation (e.g. $p = Eu/c^2$) or even some mathematical intuition (e.g. four-vectors) (Feynman, Leighton and Sands 1963; Taylor and Wheeler 1966; Mermin 1989; Sonego and Pin 2005).

interval (δx, δt) depend only on that interval and not on the location of its end points (in the considered inertial frame). In other words, the transformed interval (δx_0, δt_0) obtained through an inertial transformation $x_0 = X(x, t, v)$ and $t_0 = T(x, t, v)$ is independent of these end points. Consider an infinitesimal interval (dx, dt), for which

$$dx_0 = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial t}dt, \tag{13}$$

$$dt_0 = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial t}dt, \tag{14}$$

The above requirement then means that the coefficients of dx and dt in the above equations must be independent of x and t, and thus X and T are linear functions of x and t, which can be written as:

$$x_0 = A(v)x + B(v)t \tag{15}$$

$$t_0 = C(v)x + D(v)t \tag{16}$$

This is an important consequence of the homogeneity of space and time. Based on this result and by further resorting to isotropy of space and the principle of relativity, one can derive the Lorentz-like transformations (see, e.g. Lévy-Leblond 1976; Pal 2003 for details):

$$x_0 = \frac{x - vt}{\sqrt{1 - v^2/K^2}} \tag{17}$$

$$t_0 = \frac{t - xv/K^2}{\sqrt{1 - v^2/K^2}} \tag{18}$$

where K is an undetermined constant velocity. When $K = 0$ one obtains the Galileo transformations, while when $K > 0$ one obtains the Lorentz transformations. It can be argued the case of $K > 0$ is favored as the resulting Lorentz transformations contain a richer structure of symmetry. In the following, we will show that the Lorentz transformations also has the advantage of determining the energy-momentum relation.

Suppose the state of a free particle is $\psi = e^{i(p_0 x_0 - E_0 t_0)}$, an eigenstate of P, in S_0, where p_0, E_0 is the momentum and energy of the particle in S_0, respectively. When described in S by coordinates x, t, the state is

$$\psi = e^{i(p_0 \frac{x - vt}{\sqrt{1-v^2/c^2}} - E_0 \frac{t - xv/c^2}{\sqrt{1-v^2/c^2}})} = e^{i(\frac{p_0 + E_0 v/c^2}{\sqrt{1-v^2/c^2}}x - \frac{E_0 + p_0 v}{\sqrt{1-v^2/c^2}}t)} \tag{19}$$

This means that in frame S the state is still the eigenstate of P, and the corresponding momentum p and energy E is[7]

[7]Alternatively we can obtain the transformations of momentum and energy by directly requiring the relativistic invariance of the momentum eigenstate $e^{i(px-Et)}$, which leads to the relation $px - Et = p_0 x_0 - E_0 t_0$. Note that any superposition of momentum eigenstates is also invariant under the coordinates transformation. The reason is that it is a scalar that describes the physical state of a quantum system (except an absolute phase), and when observed in different reference frames it should be the same. This also means that the state evolution equation must be relativistically invariant. However, if the relativistically invariant equation is replaced by the nonrelativistic approximation such as the Schrödinger equation, the state will no longer satisfy the relativistic invariance.

$$p = \frac{p_0 + E_0 v/c^2}{\sqrt{1 - v^2/c^2}} \tag{20}$$

$$E = \frac{E_0 + p_0 v}{\sqrt{1 - v^2/c^2}} \tag{21}$$

We further suppose that the particle is at rest in frame S_0. Then the velocity of the particle is v in frame S[8]. Considering that the velocity of a particle in the momentum eigenstate $e^{i(px-Et)}$ or a wavepacket superposed by these eigenstates is defined as the group velocity of the wavepacket, namely

$$u = \frac{dE}{dp}, \tag{22}$$

we have

$$dE_0/dp_0 = 0 \tag{23}$$

$$dE/dp = v \tag{24}$$

Eq.(23) means that E_0 and p_0 are independent. Moreover, since the particle is at rest in S_0, E_0 and p_0 do not depend on v. By differentiating both sides of Eq.(20) and Eq.(21) relative to v we obtain

$$\frac{dp}{dv} = \frac{v}{c^2}\frac{p_0 + E_0 v/c^2}{(1 - v^2/c^2)^{\frac{3}{2}}} + \frac{E_0/c^2}{(1 - v^2/c^2)^{\frac{1}{2}}} \tag{25}$$

$$\frac{dE}{dv} = \frac{v}{c^2}\frac{E_0 + p_0 v}{(1 - v^2/c^2)^{\frac{3}{2}}} + \frac{p_0}{(1 - v^2/c^2)^{\frac{1}{2}}} \tag{26}$$

Dividing Eq.(26) by Eq.(25) and using Eq.(24) we obtain

$$\frac{p_0}{\sqrt{1 - v^2/c^2}} = 0 \tag{27}$$

This means that $p_0 = 0$. Inputing this important result to Eq.(21) and Eq.(20), we immediately have

$$E = \frac{E_0}{\sqrt{1 - v^2/c^2}}, \tag{28}$$

$$p = \frac{E_0 v/c^2}{\sqrt{1 - v^2/c^2}}, \tag{29}$$

Then the energy-momentum relation is:

$$E^2 = p^2c^2 + E_0^2 \tag{30}$$

where E_0 is the energy of the particle at rest, called rest energy of the particle, and p and E is the momemtum and energy of the particle with velocity v. By defining $m = E_0/c^2$ as the (rest) mass of the particle, we can further obtain the familar energy-momentum relation

[8] Note that we can also get this result from the definition Eq. (22) by using the above transformations of momentum and energy Eq.(20) and Eq.(21).

$$E^2 = p^2c^2 + m^2c^4 \tag{31}$$

In the nonrelativistic domain, this energy-momentum relation reduces to $E = p^2/2m$.

4 The free Schrödinger equation

The relation between energy E and momentum p in the relativistic domain implies that the operator relation is $H = P^2c^2+m^2c^4$ for an isolated system, where H is called the free Hamiltonian of the system. By inputing this operator relation to the evolution equation Eq.(5), we can obtain the free evolution equation, which assumes the same form as the free particle Klein-Gordon equation:

$$\frac{\partial^2\psi(x,t)}{\partial t^2} - c^2\frac{\partial^2\psi(x,t)}{\partial x^2} + m^2c^4\psi(x,t) = 0 \tag{32}$$

In the nonrelativistic domain the operator relation reduces to $H = P^2/2m$ for an isolated system. The corresponding free evolution equation assumes the same form as the free particle Schrödinger equation:

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} \tag{33}$$

Here it needs to be justified that the only parameter m in this equation assumes real values; otherwise the appearance of the imaginative unit i in the equation will be an illusion and the equation will be distinct from the free particle Schrödinger equation. By Eq.(29) and the definition $m = E_0/c^2$, this is equivalent to proving that p or the eigenvalue of the generator of space translation P assumes real values, namely that the generator of space translation P is Hermitian. This is indeed the case. Since the space translation operator $T(a)$ preserves the norm of the state: $\int_{-\infty}^{\infty}\psi^*(x,t)\psi(x,t)dx = \int_{-\infty}^{\infty}\psi^*(x-a,t)\psi(x-a,t)dx$, $T(a)$ is unitary, satisfying $T^\dagger(a)T(a) = I$. Thus the generator of space translation P, which is defined by $T(a) = e^{-iaP}$, is Hermitian.

In addition, it is worth noting that, unlike the free particle Schrödinger equation, the reduced Planck constant $\hbar$ with dimension of action is missing in this equation. However, this is in fact not a problem. The reason is that the dimension of $\hbar$ can be absorbed in the dimension of the mass m. For example, we can stipulate the dimensional relations as $p = 1/L$, $E = 1/T$ and $m = T/L^2$, where L and T represents the dimensions of space and time, respectively (see Duff, Okun and Veneziano 2002 for a more detailed analysis). Moreover, the value of $\hbar$ can be set to the unit of number 1 in principle. Thus the above equation is essentially the free particle Schrödinger equation in quantum mechanics.

By using the definition of classical potential and requiring an appropriate expectation value correspondence, $d < P > /dt = - < \partial V/\partial x >$, we can further obtain the Schrödinger equation under an external potential $V(x,t)$[9]:

[9] In order to derive the complete Schrödinger equation in a fundamental and strict way, we need a fundamental theory of interactions such as quantum field theory. It will be interesting to see whether the forms of basic interactions are also restricted or even determined by certain properties of space and time.

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t). \quad (34)$$

The general form of a classical potential may be $V(x, \frac{\partial}{\partial x}, t)$, and its concrete form is determined by the non-relativistic approximation of the quantum interactions involved, which are described by the relativistic quantum field theory.

5 Conclusions

The free Schrödinger equation in quantum mechanics is usually derived in textbooks by analogy and correspondence with classical mechanics (see, e.g. Schiff 1968; Landau and Lifshitz 1977; Greiner 1994)[10]. It begins with the assumption that the state of a free microscopic particle has the form of a plane wave $e^{i(kx-\omega t)}$. When combining with the de Broglie relations for momentum and energy $p = \hbar k$ and $E = \hbar\omega$, this state becomes $e^{i(px-Et)/\hbar}$. Then it uses the nonrelativistic energy-momentum relation $E = p^2/2m$ to obtain the free particle Schrödinger equation. There are at least two mysteries in such a heuristic derivation. First, even if the behavior of microscopic particles likes wave and thus a wave function is needed to describe them, it is unclear why the wave function must assume a complex form. Indeed, when Schrödinger originally invented his equation, he was also puzzled by the inevitable appearance of the imaginary unit "i" in the equation. Next, one doesn't know why there are the de Broglie relations for momentum and energy and why the nonrelativistic energy-momentum relation is $E = p^2/2m$. Usually one can only resort to experience and classical physics to answer these questions. This seems unsatisfactory because quantum mechanics is generally regarded as a more fundamental theory, of which classical mechanics is only an approximation.

According to the above analysis, the key to unveil these mysteries is to analyze the homogeneity of space and time and the resulting spacetime translation invariance of natural laws. Spacetime translation gives the definitions of momentum and energy in quantum mechanics. The momentum operator P is defined as the generator of space translation, and it is Hermitian and its eigenvalues are real. Moreover, the momentum operator can be uniquely determined by its definition, which turns out to be $P = -i\frac{\partial}{\partial x}$, and its eigenstate is e^{ipx} , where p is the real eigenvalue. Similarly, the energy operator H is defined as the generator of time translation. But its form is determined by the concrete situation. Fortunately, for an isolated system (e.g. a free microscopic particle) the form of energy operator, which determines the evolution equation, can be fixed for linear evolution by the requirements of spacetime translation invariance and relativitic invariance. Concretely speaking, time translational invariance requires that $dH/dt = 0$, and thus the solutions of the evolution equation $i\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t)$ assume the form $\psi(x,t) = \varphi_E(x)e^{-iEt}$. Moreover, space translational invariance requires $[P, H] = 0$, and this further determines

[10]There are also some attempts to derive the Schrödinger equation from Newtonian mechanics, one typical example of which is Nelson's stochastic mechanics (Nelson 1966). However, it has been argued that Nelson's derivation is problematic, and in particular, stochastic mechanics is inconsistent with quantum mechanics (Grabert, Hänggi, and Talkner 1979; Wallstrom 1994). In fact, Nelson himself also showed that there is an empirical difference between the predictions of quantum mechanics and his stochastic mechanics when considering quantum entanglement and nonlocality (Nelson 2005).

that $\varphi_E(x)$ is the eigenstate of P, namely $\varphi_E(x) = e^{ipx}$. Therefore, spacetime translation invariance entails that the state of a free particle with definite momentum and energy assumes the plane wave form $e^{i(px-Et)}$. Furthermore, the relation between p and E or the energy-momentum relation can be determined by the relativistic invariance of the free state $e^{i(px-Et)}$, and its nonrelativistic approximation is $E = p^2/2m$. Then we can obtain the energy operator for a free particle, $H = P^2/2m$, and the free particle Schrödinger equation, Eq.(33). This analysis might answer why the imaginary unit "i" appears in the wave equation and why there are the de Broglie relations and why the nonrelativistic energy-momentum relation is what it is.

In conclusion, we have argued that the free Schrödinger equation may be regarded as a consequence of spacetime translation invariance and relativistic invariance when assuming linearity of time evolution. Though the requirements of these invariances are already well known, a strict derivation of the free Schrödinger equation in terms of them seems still missing in the literature[11]. The new integrated analysis might help undertsand the origin of the wave equation in quantum mechanics.

References

[1] Brown, H. (2005). Physical Relativity: Space-time Structure from a Dynamical Perspective. Oxford: Clarendon Press.

[2] Duff, M. J., Okun, L. B., and Veneziano, G. (2002). Trialogue on the number of fundamental constants. Journal of High Energy Physics, 03, 023.

[3] Feynman, R. P., Leighton, R. B., and Sands, M. (1963). The Feynman Lectures on Physics, vol. I. Reading: Addison-Wesley, pp. 16/6-16/7.

[4] Grabert, H., Hänggi, P., and Talkner, P. (1979). Is quantum mechanics equivalent to a classical stochastic process? Phys. Rev. A 19, 2440-2445.

[5] Greiner, W. (1994). Quantum Mechanics: An Introduction. New York: Springer.

[6] Landau, L. and Lifshitz, E. (1977). Quantum Mechanics. Oxford: Pergamon Press.

[7] Lévy-Leblond, J. M. (1967). Nonrelativistic particles and wave equations. Commun. Math. Phys. 6, 286.

[8] Lévy-Leblond, J. M. (1976). One more derivation of the Lorentz transformation. Am. J. Phys. 44, 271-277.

[9] Mermin, N. D. (1968). Space and Time in Special Relativity. New York: McGraw-Hill.

[11]Note that some authors have derived the free Schrdinger equation in terms of Galilean invariance and a few other assumptions (Lévy-Leblond 1967; Musielak and Fry 2009), and these derivatrions are different from that given here.

[10] Musielak, Z. E. and Fry J. L. (2009). Physical theories in Galilean spacetime and the origin of Schrdinger-like equations. Annals of Physics 324, 296-308.

[11] Nelson, E. (1966). Derivation of the Schrödinger equation from Newtonian mechanics. Phys. Rev. 150, 1079-1085.

[12] Nelson, E. (2005). The mystery of stochastic mechanics, manuscript 2005-11-22.

[13] Pal, P. B. (2003). Nothing but relativity. Eur. J. Phys. 24, 315-319.

[14] Schiff, L. (1968). Quantum Mechanics. New York: McGraw-Hill.

[15] Shankar, R. (1994). Principles of Quantum Mechanics, 2nd ed. New York: Plenum.

[16] Sonego, S. and Pin, M. (2005). Deriving relativistic momentum and energy. Eur. J. Phys. 26, 33-45.

[17] Taylor, E. F. and Wheeler, J. A. (1966). Spacetime Physics. New York: Freeman, pp. 106- 108.

[18] Torretti, R. (1983). Relativity and Geometry. Oxford: Pergamon Press.

[19] Wallstrom, T. (1994). Inequivalence between the Schrdinger equation and the Madelung hydrodynamic equations. Phys. Rev. A 49, 1613-1617.