The Knowability Paradox in the light of a Logic for Pragmatics

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Abstract The Knowability Paradox is a logical argument showing that if all truths are knowable in principle, then all truths are, in fact, known. Many strategies have been suggested in order to avoid the paradoxical conclusion. A family of solutions – called *logical revision* – has been proposed to solve the paradox, revising the logic underneath, with an intuitionistic revision included. In this paper, we focus on so-called *revisionary solutions* to the paradox – solutions that put the blame on the underlying logic. Specifically, we analyse a possibile translation of the paradox into a modified intuitionistic fragment of a logic for pragmatics (KILP) inspired by Dalla Pozza and Garola in 1995. Our aim is to understand if KILP is a candidate for the logical revision of the paradox and to compare it with the *standard* intuitionistic solution to the paradox.

1 Introduction

Church-Fitch's Knowability Paradox shows that from the assumptions that all truths are knowable and that there is at least an unknown truth (i.e., that we are non-omniscient) follows the undesirable conclusion that all truths are known. The paradox of knowability is considered a problem especially for antirealists on truth.

An antirealist way of answering the criticisms consists in revising logic, assuming (for example) the intuitionistic logic as the right logic, thus blocking the paradox through the adoption of a revision of the logical framework in which the derivation is made.

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We take for granted that a revison of the logical framework could be considered as the right solution to the paradox. Aim of the paper is to analyse if the paradox is reproducible within a *logic for pragmatics* (LP), specifically into a modified intuitionistic fragment of a logic for pragmatics (KILP) inspired by Dalla Pozza and Garola in 1995. The basic idea of the paper is that if some epistemic aspects associated with the notion of *assertion*, which are merely implicit in some philosophical conceptions of intuitionistic logic (on this aspect see Sundholm (1997)), can be explicated in a proper way in the pragmatic language, then KILP seems to be – at least *prima facie* – as good as other logical frameworks for the solution of the knowability paradox.

The paper is divided into eight sections. Section 2 is devoted to briefly outlining the structure of the knowability paradox. In Section 3, we sketch the intuionistic solution to the paradox. Then, an analysis of the difficulties of the intuionistic solution, specifically the *Undecidedness paradox of Knowability*, is sketched in Section 4. In Section 5, LP and ILP are introduced. Section 6 deals with an analysis of the paradox in KILP. Section 7 is devoted to a comparison between our solution and the intuitionistic one. Some provisional conclusions of the paper are outlined in the last section.

2 Knowability Paradox

The *Knowability Paradox* is a proof that, if every truth is knowable, then every truth is also actually known. Such a paradox is based on two principles: the *principle of knowability* and the *principle of non-omniscience*. The principle of knowability KP can be expressed in the following way:

(KP) $\forall p(p \rightarrow \Diamond Kp)$

while non-omniscience (Non-Om) is formulated as:

(Non-Om) $\exists p(p \wedge \neg Kp)$

The expression $'Kp'$ reads"p is, has been or will be known by somebody". Assume the following two properties of knowledge:

1. the distributive property over conjunction (Dist), i.e., if a conjunction is known, then its conjuncts are also known, and
2. the factivity of knowledge (Fact), i.e., if a proposition is known, then it is true.

Assume the following two unremarkable modal claims, which can be formulated using the usual modal operators $\Diamond$ ("it is possible that") and $\Box$ ("it is necessary that"). The first is the *Rule of Necessitation*:

(Nec) If p is a theorem then $\Box p$

The second rule establishes the interdefinability of the modal concepts of necessity and possibility:

(ER) $\Box\neg p$ is logically equivalent to $\neg\Diamond p$

From KP and Non-Om a contradiction follows. Fitch (1963) and Church (we follow here Salerno 2009) proved that

(*) $\forall p \neg\Diamond K(p \wedge \neg Kp)$

is a theorem. But if (*) and (Non-Om) hold, then (KP) has to be rejected, since the substitution of $p \wedge\neg Kp$ for p in (KP) leads to a contradiction.

On the other hand, if (KP) is accepted, then (Non-Om) must be denied. However, the negation of (Non-Om) is equivalent to the formula asserting that "$\forall p(p \rightarrow Kp)$".

Therefore, from (KP) it follows that every sentence is known and this fact seems to be particularly problematic for the holders of antirealism who accept (KP). This argumentation shows that in the presence of (relatively unproblematic) principles (Dist) and (Fact), the thesis that all truths are knowable (KP) entails that all truths are known. Since the latter thesis is clearly unacceptable, the former must be rejected. We must conclude conceding that some truths are unknowable.

The proof of the theorem is based on the two following arguments that hold in any minimal modal system.

First argument:

(1) $p \wedge\neg Kp$	instance of Non-Om
(2) $(p \wedge\neg Kp) \rightarrow \Diamond K(p \wedge\neg Kp)$	substitution of "$p \wedge\neg Kp$" for p in KP
(3) $\Diamond K(p \wedge\neg Kp)$	from (1) and (2) and *Modus Ponens*

Second independent argument:

(4)	$K(p \wedge \neg Kp)$	assumption
(5)	$(Kp \wedge K\neg Kp)$	distributivity of K
(6)	$(Kp \wedge \neg Kp)$	factivity of K
(7)	$\bot$	contradiction
(8)	$\neg(K(p \wedge \neg Kp))$	*reductio*, discarging (4)
(9)	$\Box\neg(K(p \wedge \neg Kp))$	(Nec)
(10)	$\neg\Diamond(K(p \wedge \neg Kp))$	(ER)

From (3) and (10) a contradiction follows. The result of the paradox can be summarized in the following theorem:

(T1) $\exists q(q \wedge \neg Kq) \rightarrow \neg\forall q(q \rightarrow \Diamond Kq)$

Furthermore, notice also that the converse of (T1) can be easily demonstrated; in fact, by the principle that what is actual is possible, we obtain the theorem:

(T2) $\forall q(q \rightarrow Kq) \rightarrow \forall q(q \rightarrow \Diamond Kq)$.

which is provably equivalent to the theorem:

(T3) $\neg\forall q(q \to \Diamond Kq) \to \exists q(q \wedge \neg Kq)$

(T1) and (T3) validate the following theorem:

(T) $\exists q(q \wedge \neg Kq) \leftrightarrow \neg\forall q(q \to \Diamond Kq))$.

If (T) is a theorem, by applying the *Rule of Necessitation* to (T), we obtain:

(TN) $\Box(\exists q(q \wedge \neg Kq) \leftrightarrow \neg\forall q(q \to \Diamond Kq))$.

Now, notice that (Non-Om) $\exists p(p \wedge \neg Kp)$ – the non-omniscience thesis – is the result of a commonsensical observation according to which, *de facto*, actually there are true propositions that we do not know. It is not a logical principle of the paradox, nor it is introduced through a logical argument.

3 The revision of the logical framework: on the Intuitionistic Solution to the Knowability Paradox

Different ways to block the knowability paradox have been proposed. They are usually grouped into three main categories:

- Restriction of the possible instances of KP.
- Reformulation of the formalization of the knowability principle.
- Revision of the logical framework in which the derivation is made.

As mentioned, we only concentrate on the last set of proposals, specifically on the intuitionistic proposal of revising the logical framework. Intuitionistic logic is considered as the right logic in an antirealistic conception of truth, a conception embracing an *epistemic* point of view on truth. A version of this epistemic conception, compatible with intuitionism, is the following one:

(A) A is true if and only if it is possible to exibit a direct justification for A.

If a justification is something connected to our linguistic capacities, namely not transcending our epistemic capacities, an antirealist can infer that:

(B) If it is possible to exhibit a direct justification for A, then it is possible to know that A.

Putting (A) and (B) together we get the knowability principle:

(KP) If A is true, then it is possible to know that A.

But, as said, from KP, every sentence turns out to be known. Supporters of an intuitionistic solution to the knowability paradox argue that

(KP) If A is true, then it is possible to know that A.

can be weakened and formulated as a valid intuitionistic formula:

(KPI) $\forall p\,(p \rightarrow \neg\neg Kp)$

obtaining in this way a formula blocking the paradox (Williamson 1982). Indeed, consider the conclusion of the paradox, i.e.:

$\neg\exists p(p \wedge \neg Kp)$.

From the conclusion we may intuitionistically derive

$\forall p \neg(p \wedge \neg Kp)$.

But if the double negation is not eliminated, then an instance of the above formula:

$\neg(p \wedge \neg Kp)$

does not entail

$(p \rightarrow Kp)$.

It only entails (KPI). An anti-realist is ready to accept (KPI), provided that the logical constants are understood in accordance with intuitionistic rather than classical logic. Following Dummett (2009), an anti-realist will prefer (KPI) to (KP) as a formalization of his view concerning the relation of truth to knowledge.

4 Difficulties in the Intuitionistic Solution to the Knowability Paradox

There are two connected difficulties regarding the intuitionistic revision of the logic for the treatment of the *knowability paradox.*

Firstly, according to Dummett (2009), the consequent of KPI means, from an intuitionistic point of view, that “there is an obstacle in principle to our being able to deny that *p* will ever be known”, or, in other words “the possibility that *p* will come to be known always remains open”. From an anti-realistic point of view, the last claim holds good for every propositions *p*. In Dummett’s opinion this is what (KPI) expresses. Observe that anti-realists (or justificationists) do not deny that there are true proposition that *in fact* will never be known,

> ... But that there are true propositions that are *intrinsically* unknowable: for instance one stating the exact mass in grams, given by a real number, of the spanner I am holding in my hand. (Dummett 2009, p. 52)

Now, although intrinsically unknowable propositions are difficult to be thought, one may consider the following sentence due to Arthur Pap (1962) as a possible objection to Dummett's thesis (a similar sentence can be found in Poincaré's works[1]):

> Every body in the universe, including our measuring rods, is constantly expanding, the rate of expansion being exactly the same for all bodies" (p. 37)[2].

Pap's sentence is not verifiable, even if it has a definite truth-condition; namely we know how the world should be in order to make the sentence true. This point was also envisaged by Russell (in Schilpp 1951)[3]. If we accept such analysis of Pap's sentence we obtain a case where it does not happen that it is possible to known a certain sentence *p*, even if we know its truth-conditions.

Let us focus on the intuitionistic revision proposed by Dummett (2009) and Williamson (1982). Is their solution satisfactory? Marton (2006, p. 86) observes that to answer this question, one should notice that any *verificationist* theory should include empirical propositions. So, Marton reformulates the question in the following way:

> Can Williamson's solution be extended to empirical propositions? This is certainly a highly problematic question, as Williamson repeatedly emphasized (e.g., 1994, 135-137), the intuitionistic approach to the paradox can only work if the intuitionistic semantics is also granted. No such generally accepted semantics of empirical propositions seems to be available, however.

This same fact was already pointed out by Prawitz (2002) when he observed that the serious obstacles to the project of generalizing a verificationist theory to empirical discourse concern sentences for which there are no conclusive verifications (2002, p. 90). Thus, if knowability is an essential feature of the antirealist paradigm in philosophy, when applying antirealist theses to empirical sentences, things become at least complex. Mathematical truths are necessary, while empirical truths can be contingent and this is considered a problem for the antirealist thesis, since empirical sentences can hardly be proven conclusively, and sometimes not just *de facto* but because they are *intrinsically* unknowable [4]. Thus, it seems that the an-

[1] See (Poincaré 1914), section II.1.

[2] An interesting analysis of the issue can be found in Dalla Pozza (2008).

[3] "My argument for the law of excluded middle and against the definition of "truth" in terms of "verifiability" is not that it is impossible to construct a system on this basis, but rather that it is possible to construct a system on the opposite basis, and that this wider system, which embraces unverifiable truths, is necessary for the interpretation of beliefs which none of us, if we were sincere, are prepared to abandon" (p. 682).

[4] Dag Prawitz (2012) points out that empirical and mathematical assertions can be justified by means of different grounds. He remarks that "a ground for the assertion of a numerical identity would be obtained by making a certain calculation, and outside of mathematics, a ground for asserting an observational sentence would be got by making an adequate observation". Dummett (2004), in fact, points out that: "The intuitionist theory of meaning applies only to mathematical

tirealist notion of truth cannot be easily associated with knowability in the case of empirical statements, since empirical sentences may be not decidable[5].

A second problem for the antirealist concerns *undecidedness*: a stronger knowability paradox named *undecidedness paradox* is derivable from the intuitionistic revision. Percival (1990) argues that the intuitionistic revision of the paradox involves a further paradox stating that there are no necessary undecided statements, which seems absurd also from the verificationist perspective. Consider the assumption that there are undecided statements in the intuitionistic and epistemic calculus:

(1) $\exists p(\neg Kp \wedge \neg K\neg p)$	Assumption (undecidedness)
(2) $(\neg Kp \wedge \neg K\neg p)$	from (1); instantiation
(3) $\forall p(\neg Kp \rightarrow \neg p)$	intuitionistically equivalent to the denial of Non-Om
(4) $(\neg Kp \rightarrow \neg p)$	instantiation of (3)
(5) $(\neg K\neg p \rightarrow \neg\neg p)$	substitution of p with $\neg p$
(6) $\neg p \wedge \neg\neg p$	contradiction from (2), (4) and (5)
(7) $\neg\exists p(\neg Kp \wedge \neg K\neg p)$	from (1) and (6)

In the above argument an intuitionistic contradiction follows. Thus, the antirealist using intuitionistic logic cannot hold that there are undecided statements and this seems absurd. A possible way to escape the conclusion is to use Williamson's strategy by formalizing undecidedness as:

$$\neg\forall p(Kp \vee K\neg p).$$

The above is classically, but not intuitionistically, equivalent to (1). So, it is only classically, but not intuitionistically, inconsistent with the result at line (6).

Has the logic of pragmatics LP some good points when handling the above problems?

5 An outline of the Logic for Pragmatics LP

Dalla Pozza & Garola in 1995 proposed a pragmatic interpretation of intuitionistic propositional logic as a *logic of assertions*. They were mainly inspired by the logics of Frege and Dummett and by Austin's theory of illocutory acts.

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statements, whereas a justificationist theory is intended to apply to the language as a whole. The fundamental difference between the two lies in the fact that, whereas a means of deciding a range of mathematical statements, or any other effective mathematical procedure, if available at all, is permanently available, the opportunity to decide whether or not an empirical statement holds good may be lost: what can be effectively decided now will no longer be effectively decidable next year, nor, perhaps, next week" (p. 42).

[5] See also Hand (2010).

Roughly speaking, the idea is to follow Frege distinguishing propositions from judgements. To briefly recapitulate Frege's distinction: the proposition has a truth value, while a judgement is the acknowledgement of the truth by a proposition. Propositions can be either true or false, while the act of judgement can be expressed through an act of assertion, which can be justified (hereafter *"J"*) or unjustified (hereafter *"U"*).

The idea of a pragmatic analysis of sentences/propositions has been developed by Reichenbach (1947). Following Frege and Reichenbach, in Dalla Pozza and Garola the assertion sign $\vdash$ consists of two parts: the horizontal stroke is a sign showing that the content is judgeable, the vertical stroke is a sign showing that the propositional content is asserted [6]. Differently from Frege's logical system, where assertive sentences cannot be nested, in Dalla Pozza and Garola's system pragmatic connectives are introduced to build complex formulas out of expressions of assertion.

Moreover, following Reichenbach's observations on assertions, i.e. that (i) assertions are part of the pragmatic aspects of language and (ii) assertions cannot be connected with truth-functional operators, in LP there are two sets of formulas: *radical* and *sentential* formulas. Every sentential formula contains at least a radical formula as a proper subformula. Radical formulas are semantically interpreted by assigning them with a (classical) truth value, while sentential formulas are pragmatically evaluated by assigning them a justification value (*J, U*), defined in terms of the intuitive notion of proof. Assertive connectives have a meaning which is explicated by the *BHK* (Brouwer, Heyting, Kolmogorov) intended interpretation of logical constants. Namely, atomic formulas are justified by a proof, while the justification of an implication is a method transforming a justification of the antecedent into a justification of the consequent, and so on.

The pragmatic language LP is the union of the set of radical formulas RAD and the set of sentential formulas SENT, which can be recursively defined:

RAD $\gamma ::= p; \neg\gamma; \gamma_1 \wedge \gamma_2; \gamma_1 \vee \gamma_2; \gamma_1 \rightarrow \gamma_2; \gamma_1 \leftrightarrow \gamma_2.$

SENT (i) atomic assertive: $\eta ::= \vdash \gamma$

(ii) Assertive $\delta ::= \eta; \sim \delta; \delta_1 \cap \delta_2; \delta_1 \cup \delta_2; \delta_1 \supset \delta_2; \delta_1 \equiv \delta_2.$

As proved by Dalla Pozza & Garola (1995), classical logic is expressed in LP by means of those valid pragmatic assertions that are elementary (i.e., the sentential formulas that do not include pragmatic connectives). This classical fragment is called (CLP). In this way, the corresponding radical formulas are tautological molecular expressions. On the other hand, intuitionistic logic is obtained by limiting the language of LP to complex formulas that are valid with atomic radical, even if the metalanguage is still classical. This intuitionistic fragment is called ILP.

The semantic rules for radical formulas are the usual Tarskian rules that specify the truth-conditions by means of a semantic assignment-function σ. Let γ_1, γ_2 be radical formulas, then:

[6] From this perspective, notice that an assertion is a "purely logical entity" independent of the speaker's intentions and beliefs.

- (i) $\sigma(\neg\gamma_1) = 1$ iff $\sigma(\gamma_1) = 0$
- (ii) $\sigma(\gamma_1 \wedge \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ and $\sigma(\gamma_2) = 1$
- (iii) $\sigma(\gamma_1 \vee \gamma_2) = 1$ iff $\sigma(\gamma_1 = 1)$ or $\sigma(\gamma_2) = 1$
- (iv) $\sigma(\gamma_1 \rightarrow \gamma_2) = 1$ iff $\sigma(\gamma_1) = 0$ or $\sigma(\gamma_2) = 1$

There are also justification rules formalized by the pragmatic evaluation π governing the justification-conditions for assertive formulas in function of the σ assignments of truth-values for the radical atomic formulas (namely, π depends on the semantic function σ for radical atomic formulas). A pragmatic evaluation function is such that

$$\pi : \delta \in EN \longmapsto_{\pi} \delta \in \{J, U\}$$

Proposition 1. *Let γ be a radical formula. Then, $\pi(\vdash \gamma) = J$ iff there is a proof that γ is true, i.e. σ assigns to γ the value "true". Hence, $\pi(\vdash \gamma) = U$ iff no proof exists that γ is true.*

Proposition 2. *Let δ be a sentential formula. Then, $\pi(\sim \delta) = J$ iff a proof exists that δ is unjustified, namely that $\pi(\delta) = U$.*

Proposition 3. *Let δ_1, δ_2 be sentential formulas, then:*

- *$\pi(\delta_1 \cap \delta_2) = J$ iff $\pi(\delta_1) = J$ and $(\delta_2) = J$*
- *$\pi(\delta_1 \cup \delta_2) = J$ iff $\pi(\delta_1) = J$ or $(\delta_2) = J$*
- *$\pi(\delta_1 \supset \delta_2) = J$ iff a proof exists that $\pi(\delta_2) = J$ whenever $(\delta_1) = J$*
- *$\pi(\delta_1 \equiv \delta_2)) = J$ iff $\pi(\delta_1 \supset \delta_2) = J$ and $\pi(\delta_2 \supset \delta_2) = J$*

Proposition 4. *Let $\gamma \in RAD$. If $\pi(\vdash \gamma) = J$ then $\sigma(\gamma) = 1$*

Modus Ponens rule is provided for both (CLP) and ILP, respectively

[MPP] If $\vdash \gamma_1, \vdash \gamma_1 \rightarrow \gamma_2$ then $\vdash \gamma_2$

and

[MPP'] If $\delta_1, \delta_1 \supset \delta_2$ then δ_2

where δ_1 and δ_2 contain only atomic radicals. Moreover, note that the justification rules do not always allow for the determination of the justification value of a complex sentential formula when all the justification values of its components are known. For instance, $\pi(\delta) = J$ implies $\pi(\sim \delta) = U$, while $\pi(\delta) = U$ does not necessary imply $\pi(\sim \delta) = J$.

In addition, a formula δ is pragmatically valid or p.valid (respectively invalid or p.invalid) if for every π and σ, the formula $\delta = J$ (respectively $\delta = U$). Note that if δ is p.valid, then $\sim \delta$ is p.invalid and if $\sim \delta$ is p.valid then δ is p.invalid. This is the criterion of validity for the pragmatic negation. We insert them just for completeness of exposition but we will not make use of them here, the same as for other pragmatic criteria of validity presented in (Dalla Pozza and Garola 1995).

Hence, no principle analogous to the truth-functionality principle for classical connectives holds for the pragmatic connectives in LP, since pragmatic connectives are partial functions of justification.

The set of radical formulas correspond to propositional formulas of classical logic, while the set of sentential formulas is obtained by applying the sign of assertion $\vdash$ to radical formulas. An assertion is justified by means of a proof and it cannot be iterated: so $\vdash\vdash \gamma$ is not a wff of LP. Nonetheless $\vdash \Box\gamma$, with $\Box$ in a S4 modality, is a wff of an extended pragmatic language with modal operators in the radical formulas. We will follow this suggestion when we will introduce the modal and epistemic operators in the intuitionistic fragment of LP. This fragment ILP is obtained limiting LP to complex formula valid with radical atomic formula. The axiom of ILP are:

A1. $\delta_1 \supset (\delta_2 \supset \delta_1)$
A2. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\delta_2 \supset \delta_3)) \supset (\delta_1 \supset \delta_2))$
A3. $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$
A4. $(\delta_1 \cap \delta_2) \supset \delta_1$; $(\delta_1 \cap \delta_2) \supset \delta_2$
A5. $\delta_1 \supset (\delta_1 \cup \delta_2)$; $\delta_2 \supset (\delta_1 \cup \delta_2)$
A6. $(\delta_1 \supset \delta_3) \supset (\delta_2 \supset \delta_3) \supset (\delta_1 \cup \delta_2) \supset \delta_3))$
A7. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\sim \delta_2)) \supset (\sim \delta_1))$
A8. $\delta_1 \supset ((\sim \delta_1) \supset \delta_2)$

The assertion sign is not a predicate and asserted sentences cannot be embedded, for instance, in the antecedent of an implication. As observed, this is a classical feature of assertion and it is what Geach (1965) calls *Frege's point*. Moreover, an assertion sign cannot be within the scope of a classical (truth conditional) connective, since it works in what is called "pragmatic capacity" (Reichenbach 1947).

Sentential formulas have an intuitionistic-like behaviour and can be translated into modal system S4, where $\vdash \gamma$ can be translated as $\Box\gamma$, meaning that "there is an (intuitive) proof of the truth of γ" in the sense of empirical or logical procedures of proof.

Briefly put, the modal meaning of pragmatic assertions is provided by the following semantic translation of pragmatic connectives. Sentential formulas can be translated into the classical modal system **S4** as in the following table:

$(\vdash \gamma)$ / $\Box\gamma$
$\sim\vdash \gamma$ / $\Box\neg\Box\gamma$
$(\vdash \gamma_1 \cap \vdash \gamma_2)$ / $\Box(\gamma_1) \wedge \Box(\gamma_2)$
$(\vdash \gamma_1 \cup \vdash \gamma_2)$ / $\Box(\gamma_1) \vee \Box(\gamma_2)$
$(\vdash \gamma_1 \supset \vdash \gamma_2)$ / $\Box(\Box(\gamma_1) \rightarrow \Box(\gamma_2))$

Classical and intuitionistic formulas are related by means of the following "bridge principles":

- (*a.*) $\vdash(\neg\gamma)\supset\sim\vdash(\gamma)$
- (*b.*) $(\vdash\gamma_1\cap\vdash\gamma_2)\equiv\vdash(\gamma_1\wedge\gamma_2)$
- (*c.*) $(\vdash\gamma_1\cup\vdash\gamma_2)\supset\vdash(\gamma_1\vee\gamma_2)$
- (*d.*) $(\vdash\gamma_1\rightarrow\gamma_2)\supset(\vdash\gamma_1\supset\vdash\gamma_2)$

The formula (*a.*) states that from the assertion of not-γ, the non-assertability of γ can be inferred. (*b.*) states that the conjunction of two assertions is equivalent to the assertion of a conjunction; (*c.*) states that from the disjunction of two assertions one can infer the assertion of a disjunction. Finally (*d.*) expresses the idea that from the assertion of a classical material implication follows the pragmatic implication between two assertions. Note that such principles hold in an extension of ILP with classical connectives. We name such fragment ILP$^+$.

6 A Pragmatic Treatment of the Knowability Paradox

Let us present the *Knowability Paradox* in the framework of ILP enriched with a knoweldge operator K and aletheic modality. Notice that such a logic cannot be ILP or ILP$^+$ because, as mentioned, intuitionistic logic is obtained by limiting the language of LP to complex formulas that are valid with atomic radical, even if the metalanguage is still classical. Given the above characterization of ILP, the formula $\vdash\Diamond Kp$ is not a wff of ILP. We extend ILP with a knoweldge operator K and modality. Concerning modality: we have already observed that $\vdash\Box\gamma$, with $\Box$ in an S4 modality, is a wff of an extended pragmatic language with modal operators in the radical formulas. Regarding the knowledge operator K: it is possible to treat it using some analogous *invariance principles* given by Ranalter in (2008) for the *ought* operator. [7] Moreover, for the sake of simplicity we will not make use of quantifiers. We start with a suitable formulation of the Knowability Principle in KILP:

(KP’): $(\vdash p\supset\vdash\Diamond Kp)$. (instance of knowability in KILP)

(KP’) is a wff of KILP and states that there exists a method transforming a proof of p into a proof of the possibility of knowledge that p, which is a stronger claim with respect to (KP), i.e. “for every p, $p\rightarrow\Diamond K\,p$”. In (KP’) one claims that there is a proof of the knowability of p. The principle of non-omniscience in KILP is – again – stronger than (Non-Om), namely:

(Non-Om’): $\vdash p\cap\sim\vdash Kp$ (instance of Non-Omniscience in KILP).

(Non-Om’) states that there is a proof of p without knowing to know that p. If so, Non-Om’ says something different form the fact it should express: i.e. non-omniscience.

[7] A similar *intermediate logic* has been developed in Bellin and Biasi (2004).

Observe that the arguments leading to the knowability paradox cannot be formulated in KILP, first of all for syntactic reasons. Let us consider the first argument:

(1’) $\vdash p \cap \sim \vdash Kp$

the substitution of “p” with “$p \wedge \neg Kp$” cannot be executed, since formulas with classical connectives are not wff of KILP. Again, the substitution of the radical formula “p” with “$\vdash p \cap \sim \vdash Kp$” in (KP’) does not work, since the sign of assertion cannot be nested. Moreover, from the substitution “$\vdash p$” with “$\vdash p \cap \sim \vdash Kp$” in (KP’), it merely follows:

(2’) $(\vdash p \cap \sim\vdash Kp) \supset \vdash \Diamond Kp$
(3’) $\vdash \Diamond Kp$ *modus ponens* from (i’) and (ii’).

Let us now consider the second independent argument of the paradox. It is worth noting that it is impossible to state the assumption for the *reductio* in KILP; namely both

(4*) $\vdash K(p \wedge \neg Kp)$
(4**) $\vdash K(\vdash p \cap \sim\vdash Kp)$

are not wff of KILP, since (4*) contains classical connectives, while in (4**) the sign of assertion is nested. Moreover, consider a semantic reading of (4*): there is a proof that we know that p is true and that we do not know that p is true. It does not make any sense! Hence, there is no way to reproduce the paradox in the language of KILP. Consequently, the argument leading to the paradox is stopped at the early inferential steps.

7 A Comparison with the Intuitionistic Solution

One could argue that the result just obtained in KILP is not surprising if KILP is an adequate extended fragment of intuitionistic logic. We have argued that in the intuitionistic solution (KP) can be weakened and formulated as a valid intuitionistic formula. Does KILP has any chances to supersede the antirealist difficulties skected in the paper, in Sec. 4? First, consider a preliminary remark. Observe that, differently from intuitionism, in KILP:

(A) *A* is true if and only if it is possible to exibit a direct justification for A

does not hold.

Indeed, for an antirealist truth is epistemically constrained, while subscribers of LP hold that what can be properly justified in LP are (assertive) acts, and propositions can be true or false. Notice, moreover, that the use of logical constants in the metalanguage of KILP is classical. That is why (A) is false in LP. In LP we have to distinguish a semantic and a pragmatic level. From the fact that a certain sentance is true it does not mean that the same sentence is justified. If (A) is false in LP then KP does not follow. In fact, KP is the result of:

(A) *A* is true if and only if it is possible to exibit a direct justification for A.

and

(B) If it is possible to exhibit a direct justication for A, then it is possible to know that A.

As already been mentioned, in putting (A) and (B) together, we get the knowability principle:

(KP) If A is true, then it is possible to know that A.

This result is in accordance with the syntactic translation given above: In KP' we have observed that we have a proof of the knowability of p whereas in KP we just claim its knowability. If KP' holds then KP holds but not *vice versa*.

Consider what happens in KILP with undecidedness.

First, observe that the argument leading to the paradox of undecidedness cannot be replicated in KILP, since we cannot even express an instance of undecidedness: Merely from a syntactical point of view

$(\sim\vdash Kp \cap \sim\vdash K\neg p)$

is not, in fact, a wff of KILP. A slightly different notion of undecidedness can be expressed by the formula:

there is a p such that $\sim(\vdash Kp \cup \sim\vdash Kp)$

namely, there is a p such that it is not provable that the assertion of Kp holds or that the assertion Kp does not hold. Let us consider the formula

there is a p such that $(\vdash p \cap \sim\vdash Kp)$

which expresses non-omniscience in KILP. An instance of the denial of non-omniscience can be now expressed in the following way:

(0) $\sim(\vdash p \cap \sim\vdash Kp)$ negation of an instance of (Non-Om').

Observe that, also with the above version of undecidenss plus Non-Om' the argument leading to the *Undecidedness Paradox of Knowability* cannot be expressed in KILP. In fact, let us consider the following steps:

(1) $\sim(\vdash Kp \cup \sim\vdash Kp)$	assumption (instance of undecidedness)
(2) $(\sim\vdash Kp \supset \sim\vdash p)$	equivalent to (0)
(3*) $(\sim\vdash K\neg p \supset \sim\vdash \neg p)$	substitution of "p" with "$\neg p$" not allowed in KILP

(1) can be assumed in order to express a stronger form of undecidability, understood as a the existence of a proof of the impossibility of obtaining decidability. Notice that the negation of the excluded middle is a contraddiction in intuitionistic logic, whereas the justification value of (1) might be undeterminate in KILP, according to the justification rules of ILP expanded to KILP. Indeed, there is a formal equivalence only among theorems of intuisionistic logic and the corresponding p.valid formula of ILP (expanded in the obvious way to KILP), while it does not follow for formulas different from theorems.

(2) can be derived and a reading of (2) – suggested by the BHK interpretation of logical constants – is the following one: there is a method which transforms a proof that Kp cannot be proven into a proof that p cannot be proven. While a classical reading of (2), namely $\neg Kp \rightarrow \neg p$, means that ignorance entails falsity, the pragmatic reading of (2) deals with the conditions of provability of K. Finally, (3*) cannot be obtained in KILP, since it is not possible to substitute "p" with "$\neg p$" (the negation is classical).

Perhaps, if one wants to express undecidedness by means of conjunction as in the original paradox, the following might do. Consider the undecidedness paradox of knowability expressed in an extension of KILP with classical negation. We name it KILP$^+$. In KILP$^+$, undecidedness can be expressed with

there is a p such that $(\sim\vdash Kp \cap \sim\vdash K\neg p)$.

Consider the following steps:

(1)' $\sim\vdash Kp \cap \sim\vdash K\neg p$	assumption (instance of undecidedness)
(2)' $(\sim\vdash Kp \supset \sim\vdash p)$	equivalent to (0)
(3)' $(\sim\vdash K\neg p \supset \sim\vdash \neg p)$	substitution of "p" with "$\neg p$" allowed in KILP$^+$
(4)' $\sim\vdash p \cap \sim\vdash \neg p$	application of the conjuncts of (1)' to (2)'and (3)'

Notice that (4)' does not involve a paradoxical consequence. The fact that we do not have a proof of p, but also we do not have a proof of $\neg p$ is rather common for empirical sentences which are not decidable.

Unlike the treatment of the undecidedness paradox of knowability in intuitionistic logic, KILP in its extension KILP$^+$ does not involve the denial of undecided sentences. So one could argue either that the paradox is not formalizable in KILP or that it is not paradoxical in one extension (KILP$^+$) of it. This seems to be an advantange of KILP over intuitionistic logic.

8 Conclusions

In this paper paper we have analysed the paradox of knowability asking if it is reproducible within a logic for pragmatics (LP), especially in an extension of an intuitionistic fragment of it, KILP. We have shown the strict limits of the proposal, but also some advantages: the most important one concerns undecideness of contingent sentences.

Notice that the negation of a sentence in intuitionistic logic means that the proposition implies the absurd and this makes sense in mathematics, while – pretheoretically – the negation of a contingent empirical proposition does not imply the absurd. On the contrary, the pragmatic negation means that there is a proof that a certain proposition is not (or cannot be) proved. The formal behaviour of the pragmatic negation can be properly understood when one take into consideration the excluded middle. It can be written as $(\vdash p \cup \sim\vdash p)$. p is an atomic formula atomic and it allows only an empirical procedure of proof; the following situation is possible: we do not have an empirical proof procedure for asserting p and we do not have any empirical procedure of proof for not asserting p. Therefore, $(\vdash p \cup \sim\vdash p)$ is not justified (see Proposition 3. (ii)).

This property of the pragmatic negation combined with the possibility to express empirical procedures of proof in the language of LP shows some possible advantanges with respect to intuitionistic logic when dealing with empirical sentences.

9 References

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