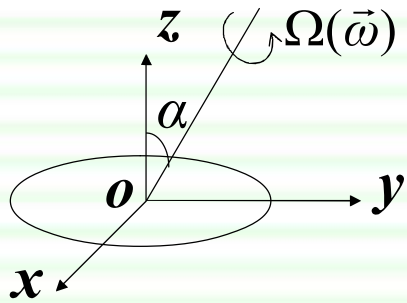
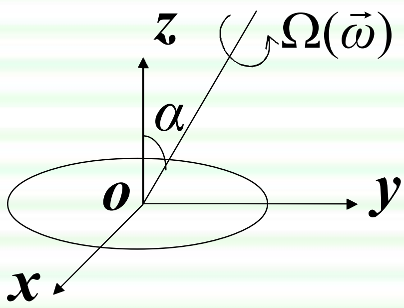
例题：一个质量为m，半径为a的圆盘绕通过质心与其垂线成α角的轴以角速度Ω转动，圆盘突然被释放，绕其质心自由转动。(1)试证明转轴在空间描绘出一个圆锥，相对于圆盘也描绘出一个圆锥。(2)试计算转轴描绘出两个圆锥分别所需要的时间。



解： （1）取动坐标系oxyz如图示：



t=0时，转轴位于yz平面内

$$\begin{cases}\omega_{y0}=\Omega\sin\alpha\\ \omega_{z0}=\Omega\cos\alpha\end{cases}$$ 即： $\omega_{x0}=0$

∵释放，则力矩M=0(题意：质心C与O重合）

由欧拉动力学方程：

$$\begin{cases}I_1\dot{\omega}_x-(I_2-I_3)\omega_y\omega_z=M_x\\ I_2\dot{\omega}_y-(I_3-I_1)\omega_z\omega_x=M_y\\ I_3\dot{\omega}_z-(I_1-I_2)\omega_x\omega_y=M_z\end{cases}$$

$$I_1=I_2=\frac{ma^2}{4} \qquad I_3=\frac{ma^2}{2}$$

$$\Rightarrow \begin{cases} \dot{\omega}_x + \omega_y \omega_z = 0 & (1) \\ \dot{\omega}_y - \omega_z \omega_x = 0 & (2) \\ \dot{\omega}_z = 0 & (3) \end{cases}$$

由(3)推出： $\omega_z = \omega_{z0} = \Omega \cos\alpha \qquad (4)$

将(2)代入求导过的(1)式中：

$$\ddot{\omega}_x + (\Omega \cos\alpha)^2 \omega_x = 0$$

$$\Rightarrow \omega_x = A \sin(\Omega \cos\alpha \cdot t) \qquad (5)$$

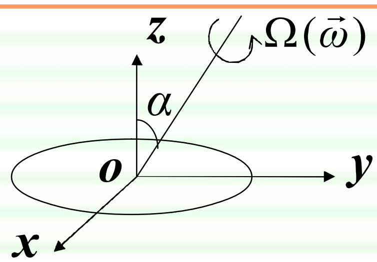
对(5)式求导： $\dot{\omega}_x = \Omega \cos\alpha A \cos(\Omega \cos\alpha \cdot t)$

代入(1)中： $\omega_y = -A \cos(\Omega \cos\alpha \cdot t)$

当t=0时， $\omega_{y0} = \Omega \sin\alpha \quad \Rightarrow A = -\Omega \sin\alpha$

$$\therefore \omega_y = \Omega \sin\alpha \cos(\Omega \cos\alpha \cdot t) \qquad (6)$$

$$\omega_z = \Omega\cos\alpha \qquad (4)$$
$$\omega_x = -\Omega\sin\alpha\sin(\Omega\cos\alpha\cdot t) \qquad (5)$$
$$\omega_y = \Omega\sin\alpha\cos(\Omega\cos\alpha\cdot t) \qquad (6)$$



$\therefore \omega = \sqrt{\omega_x^2+\omega_y^2+\omega_z^2} = \Omega$ 且ω_z也保持不变

即 ω_x, ω_y 都是以 $\Omega\cos\alpha$ 绕z轴旋转，这样它们在空间描绘出一个正圆锥。

即 $\vec{\omega}$ 绕 z 轴相对于圆盘描绘出一个正圆锥。

-----即本体极面

而 $\vec{\omega}$ 绕 z 轴一周即转轴相对于圆盘描绘出圆锥一周所需的时间为：

$$T_2 = \frac{2\pi}{\omega_z} = \frac{2\pi}{\Omega\cos\alpha}$$

（2）∵ $\omega_z = \Omega\cos\alpha \quad (4)$

$\omega_x = -\Omega\sin\alpha\sin(\Omega\cos\alpha\cdot t) \quad (5)$

$\omega_y = \Omega\sin\alpha\cos(\Omega\cos\alpha\cdot t) \quad (6)$

$$\because \vec{J} = \vec{I}\vec{\ }\cdot\vec{\omega} = I_1\omega_x\vec{i} + I_2\omega_y\vec{j} + I_3\omega_z\vec{k} = I_1(\omega_x\vec{i} + \omega_y\vec{j} + 2\omega_z)\vec{k}$$

$$\therefore \vec{\omega}\cdot\vec{J} = \vec{\omega}\cdot\vec{I}\vec{\ }\cdot\vec{\omega} = I_1(\omega_x^2 + \omega_y^2 + 2\omega_z^2)$$

$$= \frac{ma^2}{4}(\Omega^2\sin^2\alpha + 2\Omega^2\cos^2\alpha) = \frac{ma^2\Omega^2}{4}(1+\cos^2\alpha)$$

$$\therefore J = \frac{ma^2}{4}\sqrt{\Omega^2\sin^2\alpha + 4\Omega^2\cos^2\alpha} = \frac{ma^2\Omega}{4}\sqrt{1+3\cos^2\alpha}$$

$\because \vec{M} = 0 \Rightarrow \dfrac{d\vec{J}}{dt} = 0$ 所以，角动量J是恒矢量。

而 $\vec{\omega}$ 与 $\vec{J}$ 间夹角 β 也为常数。 $\because \cos\beta = \dfrac{\vec{\omega}\cdot\vec{J}}{\omega J} = \dfrac{1+\cos^2\alpha}{\sqrt{1+3\cos^2\alpha}}$

$\vec{\omega}$ 在空间围绕ζ轴也描绘出一个正圆锥---空间极面

定点转动：本体极面在空间极面上作无滑动的滚动。

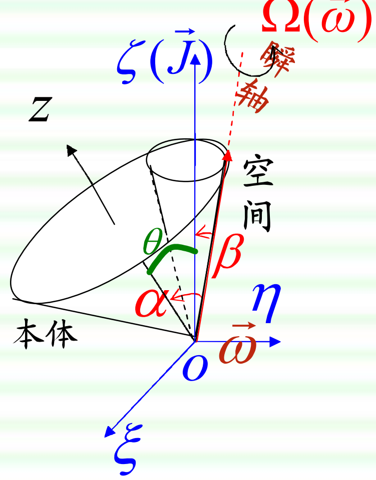
显见：θ=α - β

∵周长比=半径比 $\therefore \frac{T_1}{T_2}=\frac{2\pi\omega\sin\beta}{2\pi\omega\sin\alpha}$

$$\Rightarrow T_1=\frac{\sin\beta}{\sin\alpha}T_2=\frac{\sqrt{1-\cos^2\beta}}{\sin\alpha}T_2$$

$$=\frac{\sin\alpha\cos\alpha}{\sin\alpha\sqrt{1-\cos^2\beta}}T_2$$

$$=\frac{2\pi}{\Omega\sqrt{1+3\cos^2\alpha}}$$



$$\cos\beta=\frac{1+\cos^2\alpha}{\sqrt{1+3\cos^2\alpha}}$$

$$T_2=\frac{2\pi}{\Omega\cos\alpha}$$

T_1另解:

利用欧拉运动学方程:

$$\begin{cases} \omega_x = \dot\varphi \sin\theta \sin\psi + \dot\theta \cos\psi \\ \omega_y = \dot\varphi \sin\theta \cos\psi - \dot\theta \sin\psi \\ \omega_z = \dot\varphi \cos\theta + \dot\psi \end{cases}$$

$$\Rightarrow \omega_x^2 + \omega_y^2 = \dot\varphi^2 \sin^2\theta + \dot\theta^2$$

$$\because \theta = \alpha - \beta \Rightarrow \dot\theta = 0 \qquad \Rightarrow \omega_x^2 + \omega_y^2 = \dot\varphi^2 \sin^2\theta$$

而已知: $\omega_x^2 + \omega_y^2 = \Omega^2 \sin^2\alpha \;\Rightarrow \Omega^2 \sin^2\alpha = \dot\varphi^2 \sin^2\theta$

$$\therefore \dot\varphi = \frac{\sin\alpha}{\sin(\alpha-\beta)}\Omega = \frac{\sin\alpha}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}\Omega$$

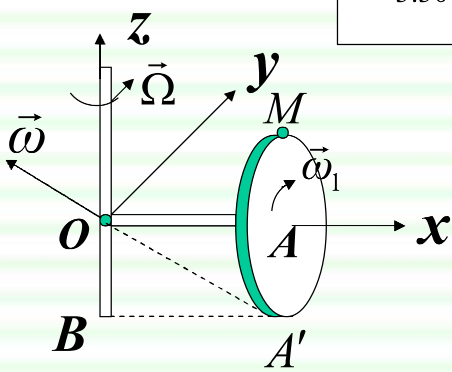
$$= \Omega\sqrt{1+3\cos^2\alpha}$$

$$\cos\beta = \frac{1+\cos^2\alpha}{\sqrt{1+3\cos^2\alpha}}$$

$$\therefore T_1 = \frac{2\pi}{|\dot\varphi|} = \frac{2\pi}{\Omega\sqrt{1+3\cos^2\alpha}}$$

$$\sin\beta = \frac{\sin\alpha\cos\alpha}{\sqrt{1+3\cos^2\alpha}}$$

【3.30】解：取动坐标系Oxyz:



据题意知：进动角速度：$\vec{\Omega} = \omega \vec{k}$

自转角速度：$\vec{\omega}_1 = \omega_1 \vec{i}$

且边缘上各点A'的速度为零？

$$\therefore (\vec{\Omega} + \vec{\omega}_1) \times \vec{r}_{A'} = 0$$

$$\vec{r}_{A'} = c\vec{i} - b\vec{j}$$

$$\Rightarrow (\omega \vec{k} + \omega_1 \vec{i}) \times (c\vec{i} - b\vec{j}) = 0$$

$$\Rightarrow c\omega \vec{j} + b\omega_1 \vec{j} = 0 \qquad \Rightarrow \vec{\omega}_1 = -\frac{c}{b}\omega \vec{i}$$

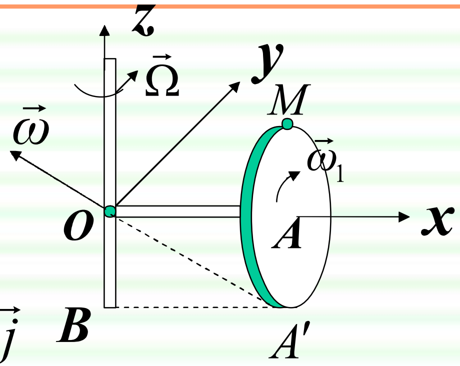
所以刚体总角速度为：$\therefore \vec{\omega} = \vec{\Omega} + \vec{\omega}_1$

$$= \omega \vec{k} - \frac{c}{b}\omega \vec{i}$$

方向沿瞬轴方向：$\vec{r}_{A'}$

连接OM，则：$\overrightarrow{oM} = \vec{r}_M = c\vec{i} + b\vec{k}$

$$\vec{\omega} = \omega\vec{k} - \frac{c}{b}\omega\vec{i}\,; \qquad \vec{r}_M = c\vec{i} + b\vec{k}$$



M点的速度:

$$\vec{v}_M = \vec{\omega}\times\vec{r}_M$$

$$= (\omega\vec{k} - \frac{c}{b}\omega\vec{i}) \times (c\vec{i} + b\vec{k}) = 2c\omega\vec{j}$$

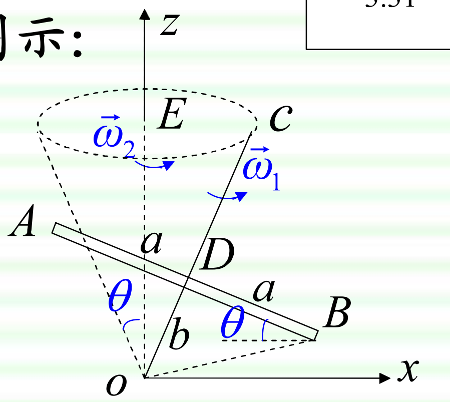
M点的加速度:

$$\vec{a}_M = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\vec{v}_M \qquad \because \begin{cases} \dot{\vec{k}} = \vec{\Omega}\times\vec{k} = \omega\vec{k}\times\vec{k} = 0 \\ \dot{\vec{i}} = \vec{\Omega}\times\vec{i} = \omega\vec{k}\times\vec{i} = \omega\vec{j} \end{cases}$$

$$= (\omega\dot{\vec{k}} - \frac{c}{b}\omega\dot{\vec{i}})(c\vec{i} + b\vec{k}) + (\omega\vec{k} - \frac{c}{b}\omega\vec{i}) \times (2c\omega\vec{j})$$

$$= -3c\omega^2\vec{i} - \frac{c^2\omega^2}{b}\vec{k} \qquad \Rightarrow |\vec{a}_M| = \omega^2 c\sqrt{9 + \frac{c^2}{b^2}}$$

【3.31】解： 取动坐标系**Oxyz**，如图示：



显见自转角速度为：$\vec{\omega}_1$

$$\vec{\omega}_1 = \omega_1 \sin\theta \vec{i} + \omega_1 \cos\theta \vec{k}$$

进动角速度为：$\vec{\Omega} = \vec{\omega}_2 = \omega_2 \vec{k}$

所以，总角速度为：

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 = \omega_1 \sin\theta \vec{i} + (\omega_1 \cos\theta + \omega_2)\vec{k}$$ 瞬轴

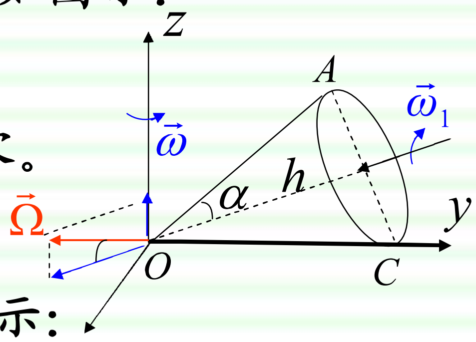
连接**OB**，则：$\vec{r}_B = \overrightarrow{oB} = \overrightarrow{oD} + \overrightarrow{DB}$

$$= (b\sin\theta \vec{i} + b\cos\theta \vec{k}) + (a\cos\theta \vec{i} - a\sin\theta \vec{k})$$

$$= (b\sin\theta + a\cos\theta)\vec{i} + (b\cos\theta - a\sin\theta)\vec{k}$$

$$\therefore \vec{v}_B = \vec{\omega} \times \vec{r}_B = \left[a\omega_1 + \omega_2 \left(a\cos\theta + a\sin\theta \right) \right] \vec{j}$$

【3.32】解：取动坐标系Oxyz，如图示：



进动角速度为：$\vec{\omega} = \omega\vec{k}$

由题意知：OC上各点的速度为零。

所以，总角速度Ω在y轴上。

设自转角速度为：$\vec{\omega}_1$ 方向如图示：

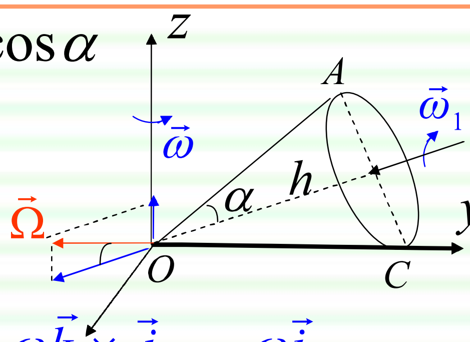
由直角三角形关系知：$\vec{\Omega} = (-\vec{j})\omega ctg\alpha$

连接OA，则：$\vec{r}_A = \overrightarrow{oA} = \dfrac{h}{\cos\alpha}(\vec{j}\cdot\cos 2\alpha + \vec{k}\cdot\sin 2\alpha)$

$$\therefore \vec{v}_A = \vec{\Omega}\times\vec{r}_A = -\vec{i}\,2h\omega\cos\alpha$$

$$\because \vec{a}_A = \frac{d\vec{\Omega}}{dt}\times\vec{r}_A + \vec{\Omega}\times\vec{v}_A = \vec{a}_1 + \vec{a}_2$$

$$\vec{\Omega}=(-\vec{j})\omega ctg\alpha; \qquad \vec{v}_A=-\vec{i}\,2h\omega\cos\alpha$$



$$\vec{r}_A=\frac{h}{\cos\alpha}(\vec{j}\cdot\cos 2\alpha+\vec{k}\cdot\sin 2\alpha)$$

$$\therefore \vec{a}_1=\frac{d\vec{\Omega}}{dt}\times\vec{r}_A \qquad \because \dot{\vec{j}}=\vec{\omega}\times\vec{j}=\omega\vec{k}\times\vec{j}=-\omega\vec{i}$$

$$=(-\dot{\vec{j}})\omega ctg\alpha\times\frac{h}{\cos\alpha}(\vec{j}\cdot\cos 2\alpha+\vec{k}\cdot\sin 2\alpha)$$

$$=\frac{h\omega^2}{\sin\alpha}(\vec{i}\cos 2\alpha-\vec{j}\sin 2\alpha) \qquad \Rightarrow |\vec{a}_1|=\frac{h\omega^2}{\sin\alpha}$$

$$\vec{a}_2=\vec{\Omega}\times\vec{v}_A=(-\vec{j})\omega ctg\alpha\times(-\vec{i})2h\omega\cos\alpha$$

$$=(-\vec{k})2h\omega^2\frac{\cos^2\alpha}{\sin\alpha} \qquad \Rightarrow |\vec{a}_2|=\frac{2h\omega^2}{\sin\alpha}\cos^2\alpha$$

【3.33】解：属于对称陀螺

由欧拉动力学方程：
$$\begin{cases} I_1\dot{\omega}_x-(I_2-I_3)\omega_y\omega_z=M_x \\ I_2\dot{\omega}_y-(I_3-I_1)\omega_z\omega_x=M_y \\ I_3\dot{\omega}_z-(I_1-I_2)\omega_x\omega_y=M_z \end{cases}$$

$I_1=I_2=2I_3$

且：$\vec{M}=0$

$$\Rightarrow\begin{cases} 2I_3\dot{\omega}_x-I_3\omega_y\omega_z=0 & (1) \\ 2I_3\dot{\omega}_y+I_3\omega_z\omega_x=0 & (2) \\ I_3\dot{\omega}_z=0 & (3) \end{cases}$$

由欧拉运动学方程：
$$\begin{cases} \omega_x=\dot{\varphi}\sin\theta\sin\psi+\dot{\theta}\cos\psi \\ \omega_y=\dot{\varphi}\sin\theta\cos\psi-\dot{\theta}\sin\psi \\ \omega_z=\dot{\varphi}\cos\theta+\dot{\psi} \end{cases}$$

$$\because \theta = 60^0 \Rightarrow \dot{\theta} = 0 \quad \Rightarrow \begin{cases} \omega_x = \dot{\varphi}\sin\theta\sin\psi & (4) \\ \omega_y = \dot{\varphi}\sin\theta\cos\psi & (5) \\ \omega_z = \dot{\varphi}\cos\theta + \dot{\psi} & (6) \end{cases}$$

因为是规则进动，所以 $\dot{\varphi}$ 是常数。

对（4）式求导： $\Rightarrow \dot{\omega}_x = \dot{\varphi}\sin\theta\cos\psi \cdot \dot{\psi}$

代入（1）中： $\Rightarrow 2\dot{\varphi}\sin\theta\cos\psi \cdot \dot{\psi} = \omega_y\omega_z \quad (7)$

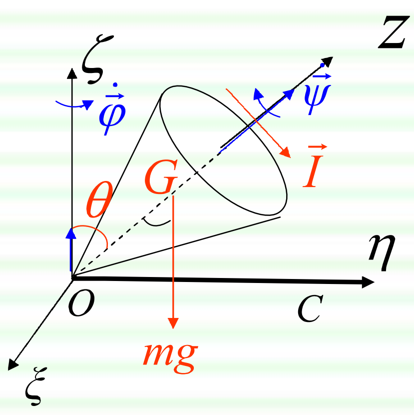
将(5)(6)代入(7)中：

$$\Rightarrow 2\dot{\varphi}\sin\theta\cos\psi \cdot \dot{\psi} = (\dot{\varphi}\sin\theta\cos\psi)(\dot{\varphi}\cos\theta + \dot{\psi})$$

$$\Rightarrow \dot{\psi} = \dot{\varphi}\cos\theta = \dot{\varphi}\cos 60^0 = \frac{\dot{\varphi}}{2}$$

已知自转角速度: $\dot{\psi} = \omega_1 \quad \therefore \omega_2 = \dot{\varphi} = 2\omega_1$

【3.35】解:



由题意知:$I_1=I_2\neq I_3$,设OG长为L.

重力矩: $\vec{M}=L\vec{k}\times mg(-\hat{\vec{\zeta}})$

而: $\hat{\vec{\zeta}}=\sin\theta\sin\psi\vec{i}+\sin\theta\cos\psi\vec{j}+\cos\theta\vec{k}$

$\therefore \vec{M}=mgL[\sin\theta\cos\psi\vec{i}-\sin\theta\sin\psi\vec{j}]$

则陀螺动力学方程为:

$$\begin{cases} I_1\dot{\omega}_x-(I_2-I_3)\omega_y\omega_z=mgL\sin\theta\cos\psi & (1)\\ I_2\dot{\omega}_y-(I_3-I_1)\omega_z\omega_x=-mgL\sin\theta\sin\psi & (2)\\ I_3\dot{\omega}_z=0 & (3) \end{cases}$$

由(3)知: $\omega_z=c$

$(1)\times\omega_x+(2)\times\omega_y+(3)\times\omega_z$:

$$\Rightarrow I_1\dot{\omega}_x\omega_x + I_2\dot{\omega}_y\omega_y + I_3\dot{\omega}_z\omega_z = mgL\sin\theta(\omega_x\cos\psi - \omega_y\sin\psi)$$

由欧拉运动学方程
$$\begin{cases} \omega_x = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_y = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_z = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases}$$

知： $\omega_x\cos\psi - \omega_y\sin\psi = \dot{\theta}$

$$\therefore I_1\dot{\omega}_x\omega_x + I_2\dot{\omega}_y\omega_y + I_3\dot{\omega}_z\omega_z = mgL\sin\theta\cdot\dot{\theta}$$

两边积分： $\Rightarrow \frac{1}{2}[I_1\omega_x^2 + I_1\omega_y^2 + I_3\omega_z^2] + mgL\cos\theta = E$

$$\Rightarrow I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + I_3\omega_z^2 = 2(E - mgL\cos\theta) \quad (4)$$

$$\because \omega_z = c \qquad \therefore \dot{\varphi}\cos\theta + \dot{\psi} = c \quad (5)$$

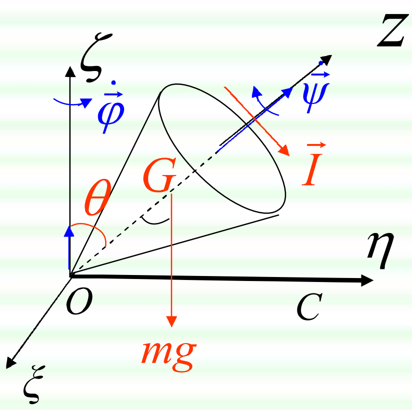
$$(4) \Rightarrow I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + I_3c^2 = 2(E - mgL\cos\theta) \quad (6)$$

$$\because mg\hat{\vec{\zeta}} \parallel \hat{\vec{\zeta}} \quad \Rightarrow \quad \vec{J}\cdot\hat{\vec{\zeta}} = \alpha$$

$$\therefore (I_1\omega_x\vec{i} + I_1\omega_y\vec{j} + I_3\omega_z\vec{k})\cdot[\sin\theta(\sin\psi\vec{i} + \cos\psi\vec{j}) + \cos\theta\vec{k}] = \alpha$$

$$\Rightarrow (I_1\omega_x \sin\theta\sin\psi + I_1\omega_y \sin\theta\cos\psi + I_3\omega_z c\text{os}\theta] = \alpha$$

$$\begin{cases} \omega_x = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_y = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_z = \dot{\varphi}\cos\theta + \dot{\psi} = c \end{cases}$$

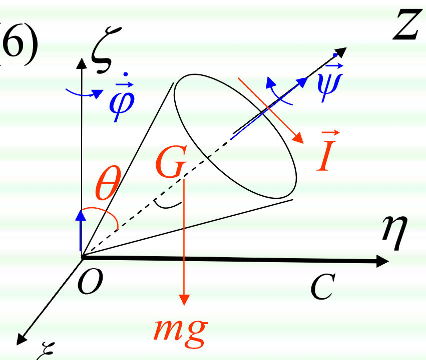


$$\Rightarrow I_1\sin^2\theta\cdot\dot{\varphi} + I_3\cos\theta\cdot c = \alpha \qquad (7)$$

(5)(6)(7)为三个第一积分.

$$\dot{\varphi}\cos\theta + \dot{\psi} = c \qquad (5)$$

$$I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + I_3c^2 = 2(E - mgL\cos\theta) \qquad (6)$$



$$I_1(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)+I_3c^2=2(E-mgL\cos\theta) \quad (6)$$

$$I_1\sin^2\theta\cdot\dot{\varphi}+I_3\cos\theta\cdot c=\alpha \quad (7)$$

$\because$ $\vec{M}\,dt = d\vec{J}$ $\quad\therefore$ $\vec{r}\times\vec{I} = \Delta\vec{J}$

即:冲量矩=角动量的增量

因为 $\vec{r}_d\times\vec{I}$ 的方向沿 ⊗与 $J_\theta(\overrightarrow{ON})$ 方向一致!

垂直于z 轴的冲量只能引起垂直于z轴的角动量的变化

$$\Rightarrow Id=I_1\dot{\theta} \quad \Rightarrow \dot{\theta}=\frac{Id}{I_1}$$

代入(6)中，当 $\theta_0=0^0$ 时,得: $2E=\dfrac{I^2d^2}{I_1}+I_3c^2+2mgL$

再代回（6）中，得:

$$I_1(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)=\frac{I^2d^2}{I_1}+2mgL(1-\cos\theta) \quad (8)$$

$$I_1 \sin^2\theta \cdot \dot{\varphi} + I_3 \cos\theta \cdot c = \alpha \qquad (7)$$

$$I_1(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) = \frac{I^2 d^2}{I_1} + 2mgL(1-\cos\theta) \qquad (8)$$

当 $t=0, \theta_0 = 0^0$ 时,代入**(7)**中,得: $I_3 c = \alpha$

再代回（7）中，得: $I_1 \dot{\varphi} \sin^2\theta + I_3 c \cos\theta = I_3 c$

$$\Rightarrow \dot{\varphi} = \frac{I_3 c(1-\cos\theta)}{I_1 \sin^2\theta}$$ 代入（8）中，得:

$$I_1[\dot{\theta}^2 + \frac{I_3^2 c^2 (1-\cos\theta)^2}{I_1^2 \sin^2\theta}] = \frac{I^2 d^2}{I_1} + 2mgL(1-\cos\theta) \qquad (9)$$

当 $\theta = \theta_{max}$ 不再变化时, $\dot{\theta} = 0$

$$\Rightarrow I_1 \frac{I_3^2 c^2 (1-\cos\theta_{max})^2}{I_1^2 \sin^2\theta_{max}} = \frac{I^2 d^2}{I_1} + 2mgL(1-\cos\theta_{max})$$

$$I_1 \frac{I_3^2 c^2 (1-\cos\theta_{\max})^2}{I_1^2 \sin^2\theta_{\max}} = \frac{I^2 d^2}{I_1} + 2mgL(1-\cos\theta_{\max})$$

利用公式:

$$1-\cos\theta_{\max} = 2\sin\frac{\theta_{\max}}{2}; \qquad \sin\theta_{\max} = 2\sin\frac{\theta_{\max}}{2}\cos\frac{\theta_{\max}}{2}$$

$$\Rightarrow tg^2(\frac{\theta_{\max}}{2}) = (\frac{Id}{I_3 c})^2 + \frac{2mgL(1-\cos\theta_{\max})I_1}{(I_3 c)^2} \approx (\frac{Id}{I_3 c})^2$$

$$\therefore \theta_{\max} = 2tg^{-1}(\frac{Id}{I_3 c})$$

而 $\dot{\varphi}\cos\theta + \dot{\psi} = c \quad (5)$

当 $\dot{\psi}$ 很大时，$\dot{\psi} \approx c = \omega_1 \quad \therefore \theta_{\max} = 2tg^{-1}(\frac{Id}{I_3\omega_1})$ 得证！

【3.36】解:

证明:本题属于有外力矩但是过重心的定点转动

由题意知:$I_1=I_2\neq I_3$,

外力矩: $\vec{M}=-I_3\lambda\vec{\omega}=-I_3\lambda(\omega_x\vec{i}+\omega_y\vec{j}+\omega_z\vec{k})$

代入欧拉动力学方程中有:

$$\begin{cases} I_1\dot{\omega}_x-(I_2-I_3)\omega_y\omega_z=-I_3\lambda\omega_x & (1) \\ I_2\dot{\omega}_y-(I_3-I_1)\omega_z\omega_x=-I_3\lambda\omega_y & (2) \\ I_3\dot{\omega}_z=-I_3\lambda\omega_z 0 & (3) \end{cases}$$

由(3)知: $\frac{d\omega_z}{dt}=-\lambda\omega_z$

$$\Rightarrow \frac{d\omega_z}{\omega_z}=-\lambda dt \quad \Rightarrow \omega_z=\Omega e^{-\lambda t} \qquad (4)$$

Ω为积分常数

$$\omega_z = \Omega e^{-\lambda t} \qquad (4)$$

将(4) 代入（1）(2)中，得:

$$\begin{cases} \dot{\omega}_x + \dfrac{I_3 - I_1}{I_1}\omega_y(\Omega e^{-\lambda t}) = -\dfrac{I_3}{I_1}\lambda\omega_x \\ \dot{\omega}_y - \dfrac{I_3 - I_1}{I_1}\omega_x(\Omega e^{-\lambda t}) = -\dfrac{I_3}{I_1}\lambda\omega_y \end{cases}$$

令: $n = \dfrac{I_3 - I_1}{I_1}\Omega$

$$\Rightarrow \begin{cases} \dot{\omega}_x + ne^{-\lambda t}\omega_y = -\dfrac{I_3}{I_1}\lambda\omega_x & (5) \\ \dot{\omega}_y - ne^{-\lambda t}\omega_x = -\dfrac{I_3}{I_1}\lambda\omega_y & (6) \end{cases}$$

$$\begin{cases} \dot{\omega}_x + ne^{-\lambda t}\omega_y = -\dfrac{I_3}{I_1}\lambda\omega_x & (5) \\ \dot{\omega}_y - ne^{-\lambda t}\omega_x = -\dfrac{I_3}{I_1}\lambda\omega_y & (6) \end{cases}$$

$i \times (5) + (6):$

$$\dot{\omega}_x i + \dot{\omega}_y + ine^{-\lambda t}\omega_y - ne^{-\lambda t}\omega_x = -\frac{I_3}{I_1}\lambda(i\omega_x + \omega_y)$$

$$\Rightarrow (\dot{\omega}_y + i\dot{\omega}_x) + ine^{-\lambda t}(\omega_y + i\omega_x) = -\frac{I_3}{I_1}\lambda(\omega_y + i\omega_x)$$

$$\therefore \frac{(\dot{\omega}_y + i\dot{\omega}_x)}{(\omega_y + i\omega_x)} = -\frac{I_3}{I_1}\lambda - ine^{-\lambda t}$$

$$\frac{(\dot{\omega}_y + i\dot{\omega}_x)}{(\omega_y + i\omega_x)} = -\frac{I_3}{I_1}\lambda - ine^{-\lambda t}$$

$$\frac{d(\omega_y + i\omega_x)}{(\omega_y + i\omega_x)} = (-\frac{I_3}{I_1}\lambda - ine^{-\lambda t})dt$$

$$\Rightarrow \ln(\omega_y + i\omega_x) = -\frac{I_3}{I_1}\lambda t + i\frac{n}{\lambda}e^{-\lambda t} + \varepsilon'$$

$$\Rightarrow (\omega_y + i\omega_x) = ae^{-\frac{I_3}{I_1}\lambda t} \cdot e^{(i\frac{n}{\lambda}e^{-\lambda t} + i\varepsilon)}$$

$$= ae^{-\frac{I_3}{I_1}\lambda t} \cdot \left[\cos(\frac{n}{\lambda}e^{-\lambda t} + \varepsilon) + i\sin(\frac{n}{\lambda}e^{-\lambda t} + \varepsilon)\right]$$

Euler公式:

$$(\omega_y + i\omega_x)$$

$$= a e^{-\frac{I_3}{I_1}\lambda t} \cdot \left[\cos\left(\frac{n}{\lambda} e^{-\lambda t} + \varepsilon\right) + i \sin\left(\frac{n}{\lambda} e^{-\lambda t} + \varepsilon\right) \right]$$

$$\therefore \begin{cases} \omega_x = a e^{-\frac{I_3}{I_1}\lambda t} \sin\left(\frac{n}{\lambda} e^{-\lambda t} + \varepsilon\right) \\ \omega_y = a e^{-\frac{I_3}{I_1}\lambda t} \cos\left(\frac{n}{\lambda} e^{-\lambda t} + \varepsilon\right) \end{cases}$$