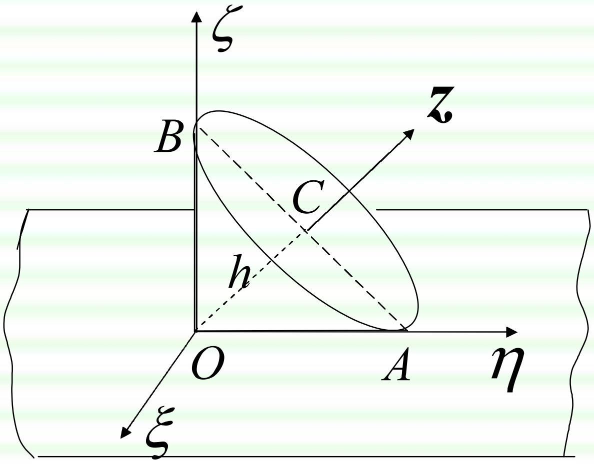
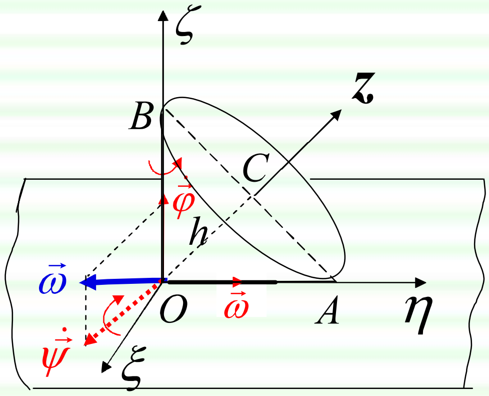
例题1、具有固定顶点O的圆锥在平面上滚动而不滑动。锥高OC=h，顶角AOB=90^0，圆锥底面中心C等速率运动，每过一秒回到原处一次。

求：在图示位置时（OA在平面上），直径AB的端点B的速度、圆锥的角加速度及A、B两点的加速度。



解：建立动坐标系 *o-xyz*

1、可通过C点求出ω的大小：



$$\therefore \vec{v}_c = \dot{\vec{\varphi}} \times \overrightarrow{oc} \quad = -\sqrt{2}\pi h \vec{e}_\xi$$

$$= 2\pi \vec{e}_\zeta \times \frac{\sqrt{2}h}{2}(\vec{e}_\eta + \vec{e}_\zeta)$$

$$\therefore \vec{v}_c = \vec{\omega} \times \overrightarrow{oc} \quad = \frac{\sqrt{2}}{2}\omega h \vec{e}_\xi$$

$$= \omega \vec{e}_\eta \times \frac{\sqrt{2}h}{2}(\vec{e}_\eta + \vec{e}_\zeta)$$

比较两式，得：$\Rightarrow \vec{\omega} = -2\pi \vec{e}_\eta$

2、$\therefore \vec{v}_B = \vec{\omega}\times\overrightarrow{oB} = -2\pi\vec{e}_\eta \times \sqrt{2}h\vec{e}_\zeta = -2\sqrt{2}\pi h\vec{e}_\xi$

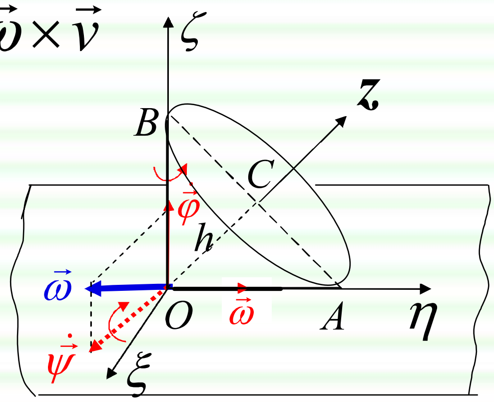
角加速度: $\vec{\beta} = \dfrac{d\vec{\omega}}{dt} = \dfrac{d(-2\pi\vec{e}_\eta)}{dt} = -2\pi\dfrac{d\vec{e}_{oA}}{dt}$ $\dot{\vec{\varphi}}\times\vec{e}_{oA}$

$= -4\pi^2\vec{e}_\zeta\times\vec{e}_{OA} = 4\pi^2\vec{e}_\xi$

线加速度: $\vec{a} = \dfrac{d\vec{v}}{dt} = \vec{\beta}\times\vec{r} + \vec{\omega}\times\vec{v}$

$\therefore \vec{r} = \overrightarrow{oA} \quad \Rightarrow \vec{a}_A$

$\therefore \vec{r} = \overrightarrow{oB} \quad \Rightarrow \vec{a}_B$



三、欧拉动力学方程:

Review:

刚体定点转动 $i=3$， 三个欧拉角描述: $\begin{cases} \varphi(t)=? \\ \theta(t)=? \\ \psi(t)=? \end{cases}$

刚体定点转动角速度沿瞬轴方向:

$$\begin{cases} \omega_x = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_y = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_z = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases}$$

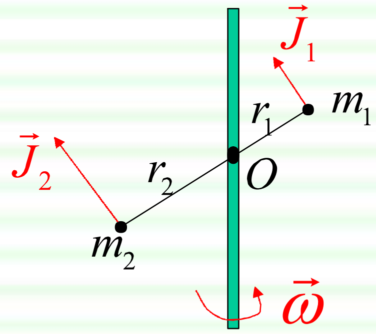
刚体定点转动加速度:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

质心运动定理: $\sum_i \vec{F}_i = m\frac{d\vec{v}_c}{dt}$

对质心的动量矩定理: $\vec{M} = \frac{d\vec{J}}{dt}$

刚体平面平行运动中: $\vec{J} = I\vec{\omega}$



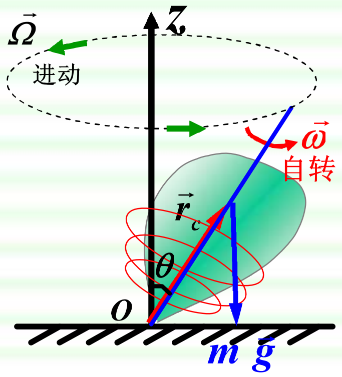
Problem: 上式是否普遍适用？否!!

反例:

图示刚体定轴转动中: $\vec{J} \nparallel \vec{\omega}$

刚体定点转动呢?

惯量张量: $\overleftrightarrow{I}$ 又一特征!



引入符号 $\mathbf{I}_{ij}$：

惯量张量：　3×3矩阵　$\overset{\rightarrow\rightarrow}{I} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$

轴转动惯量：　$\mathbf{I}_{xx}$、$\mathbf{I}_{yy}$、$\mathbf{I}_{zz}$

惯量积：　$\mathbf{I}_{xy}$

则：　①刚体角动量：　$\vec{J} = \overset{\rightarrow\rightarrow}{I}\,\vec{\omega}$

②刚体的动能：　$T = \frac{1}{2}\vec{\omega}\cdot\overset{\rightarrow\rightarrow}{I}\cdot\vec{\omega} = \frac{1}{2}I\omega^2$

$\overset{\rightarrow\rightarrow}{I} \xrightarrow{?} I$　选动系o-xyz来表述惯量张量！！！

1、可以使九个惯量系数都是常量；

2、进而，若选择惯量主轴为动系的三个坐标轴，则所有的惯量积为零；　$\overset{\rightarrow\rightarrow}{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

惯量主轴： 相关的惯量积为零 ($I_{xy}=I_{yz}=I_{zx}=0$)的轴

说明：

1、惯量张量是关于刚体定点转动的；

而转动惯量是关于轴，如：I_{xx}是各质元的质量与该质元至该轴的距离的平方的乘积之和。

$$I_{xx}=\sum_{i=1}^{n} m_i(y_i^2+z_i^2); \qquad I_{xx}=\int (y^2+z^2)dm$$

2、对同一点，惯量张量是确定的。但是坐标系选择不同，惯量系数不同。

刚体的欧拉动力学方程:

①选择原点位于刚体定点O上的动坐标系;

②取关于O的惯量主轴为动坐标轴，则: $\vec{I}^{\rightarrow} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

$$\therefore \vec{J} = I_1\omega_x\vec{i} + I_2\omega_y\vec{j} + I_3\omega_z\vec{k}$$

$$\therefore \frac{d\vec{J}}{dt} = \frac{\tilde{d}\vec{J}}{dt} + \vec{\omega}\times\vec{J} = \dot{J}_x\vec{i} + \dot{J}_y\vec{j} + \dot{J}_z\vec{k} + \vec{\omega}\times\vec{J}$$

$$= [I_1\dot{\omega}_x - (I_2 - I_3)\omega_y\omega_z]\vec{i} + [I_2\dot{\omega}_y - (I_3 - I_1)\omega_z\omega_x]\vec{j}$$

$$+ [I_3\dot{\omega}_3 - (I_1 - I_2)\omega_x\omega_y]\vec{k}$$

$$= \boldsymbol{M}_x\vec{\boldsymbol{i}} + \boldsymbol{M}_y\vec{\boldsymbol{j}} + \boldsymbol{M}_z\vec{\boldsymbol{k}}$$

则有：
$$\begin{cases} I_1\dot{\omega}_x-(I_2-I_3)\omega_y\omega_z=M_x \\ I_2\dot{\omega}_y-(I_3-I_1)\omega_z\omega_x=M_y \\ I_3\dot{\omega}_z-(I_1-I_2)\omega_x\omega_y=M_z \end{cases}$$

--------欧拉动力学方程　　**1776**年

上述推导中用了两次简化:

①、选用动坐标系可使惯性系数为常数;

②、选用**O**点的惯量主轴为动系的坐标轴，可使惯量积为零;

注意：运动是要从固定坐标系中观察的。

作业①：**3.30；3.31**

②：自学书中例题

③：概念：惯量主轴、惯量张量、

空间极面、本体极面

【1.44】题解:

1.44

利用比耐公式 $$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{F}{m}$$

$$\because F=-m\left(\frac{\mu^2}{r^2}+\frac{\nu}{r^3}\right)=-m\mu^2(u^2+\nu u^3)$$

$$\therefore h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu^2u^2+\nu u^3$$

$$\Rightarrow \mu^2+\nu u=h^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

$$\Rightarrow \frac{d^2u}{d\theta^2}+\left(1-\frac{\nu}{h^2}\right)u=\frac{\mu^2}{h^2}$$

k^2 定义已知

$$\Rightarrow \frac{d^2u}{d\theta^2}+k^2u=\frac{\mu^2}{h^2}$$

令：$u=\xi+\dfrac{\mu^2}{k^2h^2}$ $\quad\Rightarrow \dfrac{d^2\xi}{d\theta^2}+k^2\xi=0$

解为：$\xi=A\cos(k\theta-\varphi_0)$ 设：$\varphi_0=0$

则：$u=\xi+\dfrac{\mu^2}{k^2h^2}=A\cos k\theta+(\dfrac{\mu}{kh})^2$

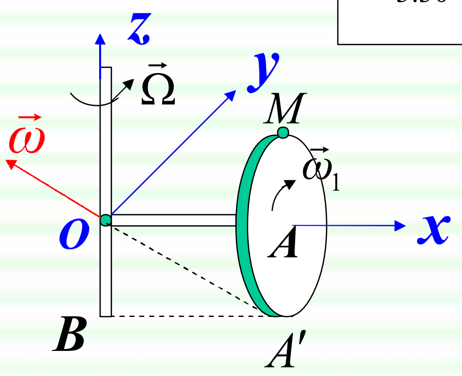
$$\therefore r=\frac{1}{u}=\frac{1}{A\cos k\theta+\dfrac{\mu^2}{h^2k^2}}=\frac{\dfrac{h^2k^2}{\mu^2}}{1+A\dfrac{h^2k^2}{\mu^2}\cos k\theta}$$

($\dfrac{h^2k^2}{\mu^2}$: a; $A\dfrac{h^2k^2}{\mu^2}$: e)

或者令：$u=\dfrac{h^2}{h^2-\nu}\xi+\dfrac{\mu^2}{h^2k^2}$ ($\dfrac{h^2}{h^2-\nu}$: $\dfrac{1}{k^2}$)

结果将一样，只是此时的$e=A\dfrac{h^2}{\mu^2}$ 原因：此A非彼A

【3.30】解：取动坐标系Oxyz:



据题意知：进动角速度：$\vec{\Omega} = \omega \vec{k}$

自转角速度：$\vec{\omega}_1 = \omega_1 \vec{i}$

且边缘上各点A'的速度为零？

$$\therefore (\vec{\Omega} + \vec{\omega}_1) \times \vec{r}_{A'} = 0$$

$$\Rightarrow (\omega \vec{k} + \omega_1 \vec{i}) \times (c\vec{i} - b\vec{k}) = 0 \qquad \vec{r}_{A'} = c\vec{i} - b\vec{k}$$

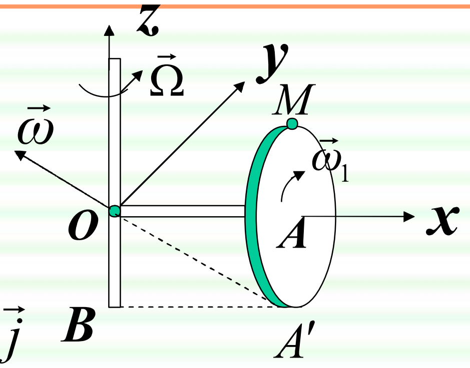
$$\Rightarrow c\omega \vec{j} + b\omega_1 \vec{j} = 0 \qquad \Rightarrow \vec{\omega}_1 = -\frac{c}{b}\omega \vec{i}$$

所以刚体总角速度为： $\therefore \vec{\omega} = \vec{\Omega} + \vec{\omega}_1$

方向沿瞬轴方向：$\vec{r}_{A'}$ $\qquad = \omega \vec{k} - \frac{c}{b}\omega \vec{i}$

连接OM，则： $\overrightarrow{oM} = \vec{r}_M = c\vec{i} + b\vec{k}$

$$\vec{\omega} = \omega\vec{k} - \frac{c}{b}\omega\vec{i}; \qquad \vec{r}_M = c\vec{i} + b\vec{k}$$



M点的速度:

$$\vec{v}_M = \vec{\omega} \times \vec{r}_M$$
$$= (\omega\vec{k} - \frac{c}{b}\omega\vec{i}) \times (c\vec{i} + b\vec{k}) = 2c\omega\vec{j}$$

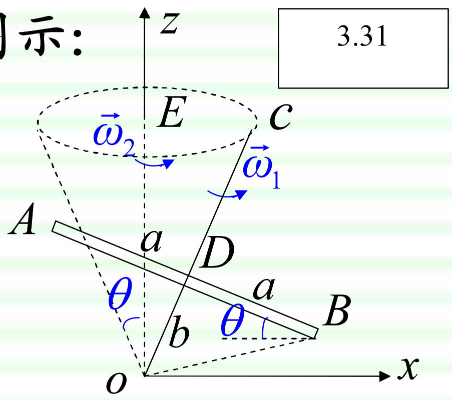
M点的加速度:

$$\vec{a}_M = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}_M$$

$$\because \begin{cases} \dot{\vec{k}} = \vec{\Omega} \times \vec{k} = \omega\vec{k} \times \vec{k} = 0 \\ \dot{\vec{i}} = \vec{\Omega} \times \vec{i} = \omega\vec{k} \times \vec{i} = \omega\vec{j} \end{cases}$$

$$= (\omega\dot{\vec{k}} - \frac{c}{b}\omega\dot{\vec{i}})(c\vec{i} + b\vec{k}) + (\omega\vec{k} - \frac{c}{b}\omega\vec{i}) \times (2c\omega\vec{j})$$
$$= -3c\omega^2\vec{i} - \frac{c^2\omega^2}{b}\vec{k} \qquad \Rightarrow |\vec{a}_M| = \omega^2 c\sqrt{9 + \frac{c^2}{b^2}}$$

【3.31】解：取动坐标系Oxyz，如图示：



显见自转角速度为：$\vec{\omega}_1$

$$\vec{\omega}_1 = \omega_1 \sin\theta \vec{i} + \omega_1 \cos\theta \vec{k}$$

进动角速度为：$\vec{\Omega} = \vec{\omega}_2 = \omega_2 \vec{k}$

所以，总角速度为：

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 = \omega_1 \sin\theta \vec{i} + (\omega_1 \cos\theta + \omega_2)\vec{k}$$ 瞬轴

连接OB，则：$\vec{r}_B = \overrightarrow{oB} = \overrightarrow{oD} + \overrightarrow{DB}$

$$= (b\sin\theta\vec{i} + b\cos\theta\vec{k}) + (a\cos\theta\vec{i} - a\sin\theta\vec{k})$$

$$= (b\sin\theta + a\cos\theta)\vec{i} + (b\cos\theta - a\sin\theta)\vec{k}$$

$$\therefore \vec{v}_B = \vec{\omega} \times \vec{r}_B = \left[a\omega_1 + \omega_2\left(a\cos\theta + a\sin\theta\right)\right]\vec{j}$$

自测题:

1、 分别用直角坐标系、图示法和角速度与速度关系方法推导平面极坐标单位矢量的导数表达式:

$$\begin{cases} \dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta \\ \dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r \end{cases}$$

并据此给出平面极坐标系下质点的运动微分方程。

2、推导毕耐公式;

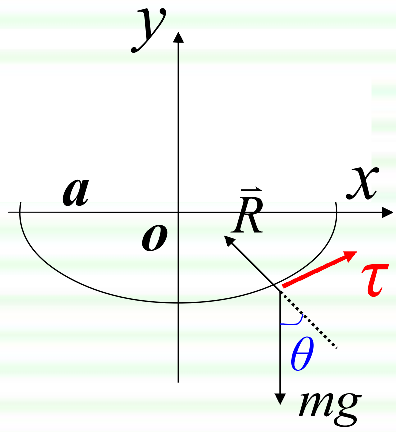
3、用毕耐公式推导散射瞄准距离与散射角的关系;

$$\rho = \frac{k'}{mv_\infty^2} ctg\frac{\varphi}{2}$$

4、掌握欧拉角的画法并指出三个欧拉角的导数方向;推导欧拉运动学方程;

【1.28】题解：解法类似例题。

采用自然坐标系的内禀方程：



$$\because \vec{F}=m\vec{a}\ \therefore \begin{cases} ma_n=\sum F_n \\ ma_\tau=\sum F_\tau \end{cases}$$

$$\Rightarrow \begin{cases} m\dfrac{v^2}{\rho}=R-mg\cdot\cos\theta & ① \\ m\dfrac{dv}{dt}=-mg\cdot\sin\theta & ② \end{cases}$$

又 $\because \dfrac{dv}{dt}=\dfrac{dv}{ds}\dfrac{ds}{dt}=v\dfrac{dv}{ds}$ 而：$\begin{cases} dy=ds\sin\theta \\ dx=ds\cos\theta \end{cases}$

所以②式：$\Rightarrow \ m v \dfrac{dv}{ds} = -m g \left(\dfrac{dy}{ds}\right)$ $\qquad \rho = \dfrac{\left(1+y'^2\right)^{3/2}}{\left|y''\right|}$

积分 $\int_0^v v dv = -\int_0^{-b} g dy$ 到顶点处速率：$v = \sqrt{2gb}$

再由①求约束反力R，显见须知ρ

$\because \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow y' = -\dfrac{b}{a^2}\dfrac{x}{\sqrt{1-(x/a)^2}}; y'' = -\dfrac{b}{a^2}\dfrac{x}{\sqrt{1-(x/a)^2}^3}$

最低点处：$\begin{cases} x_0 = 0 \\ y_0 = -b \end{cases}, \begin{cases} y' = 0 \\ y'' = -\dfrac{b}{a^2} \end{cases}$ 所以 $\rho_0 = \dfrac{a^2}{b}$

$\therefore R = m\dfrac{v^2}{\rho_0} + mg\cos\theta_0 \Big|_{\theta_0 = 0} = mg[1 + 2(\dfrac{b}{a})^2]$

最低点处对轨道的压力。

力学的解法:

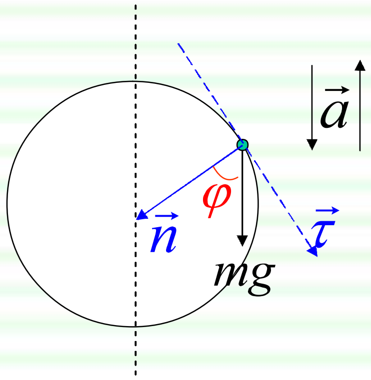
动能定理: $$mgb = \frac{1}{2}mv^2 \Rightarrow v^2 = 2gb$$

$$\because \rho_0 = \frac{a^2}{b}$$

圆周运动: $$N - mg = \frac{mv^2}{\rho} \Rightarrow N = mg(1 + \frac{2b^2}{a^2})$$

【1.33】题解：解法类似例题。

解：运动微分方程为：



$$\begin{cases} m\dfrac{dv}{dt} = mg\sin\varphi \pm ma\sin\varphi & ① \\ m\dfrac{v^2}{r} = R + mg\cos\varphi \pm ma\cos\varphi & ② \end{cases}$$

"+": $\vec{a}\downarrow$;

①式两边同乘 $v = r\dot{\varphi}$

$$\Rightarrow v\frac{dv}{dt} = (g\sin\varphi \pm a\sin\varphi)\cdot r\frac{d\varphi}{dt}$$

两边积分：$\displaystyle\int_{v_0}^{v} dv = \int_{\varphi_0}^{\varphi}(g\sin\varphi \pm a\sin\varphi)\cdot rd\varphi$

1.33

$$\Rightarrow v^2 = v_0^2 + 2(g \pm a)(\cos\varphi_0 - \cos\varphi)r$$

代入②中：

$$R = m\frac{v^2}{r} - m(g \pm a)\cos\varphi$$

$$= m\frac{v_0^2}{r} + 2m(g \pm a)(\cos\varphi_0 - \cos\varphi) - m(g \pm a)\cos\varphi$$

$$= m\frac{v_0^2}{r} + 2m(g \pm a)\cos\varphi_0 - 3m(g \pm a)\cos\varphi$$

$$= m\frac{v_0^2}{r} + mg(1 \pm \frac{a}{g})(2\cos\varphi_0 - 3\cos\varphi)$$

$"+": \vec{a}\downarrow;\qquad "-": \vec{a}\uparrow$

力学的解法:

能量守恒:

$$\frac{1}{2}mv_r^2 + m(g \pm a)r\cos\varphi = \frac{1}{2}mv_{r0}^2 + m(g \pm a)r\cos\varphi_0$$

$$\Rightarrow v_r^2 = v_{r0}^2 + 2(g \pm a)(\cos\varphi_0 - \cos\varphi)r$$

圆周运动:

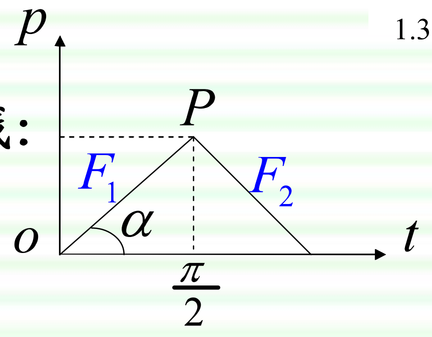
$$\frac{mv_r^2}{r} = R + m(g \pm a)\cos\varphi$$

$$\Rightarrow R = m\frac{v_r^2}{r} - m(g \pm a)\cos\varphi$$

结果一样。

【1.35】题解: 解法类似例题。

据题意画出受力随时间的变化曲线:



受力情况:

$$\begin{cases} F_1 = t \cdot tg\alpha = \dfrac{P}{0.5\tau} t = \dfrac{2P}{\tau} t \\ F_2 = (t-\tau) \cdot tg\alpha = \dfrac{P}{0.5\tau}(\tau - t) = \dfrac{2P}{\tau}(\tau - t) \end{cases}$$

由 $\vec{F} = m\dfrac{d\vec{v}}{dt}$ 得: $F = m\dfrac{dv}{dt}$

$$\therefore \begin{cases} F_1 = m\dfrac{dv}{dt} = \dfrac{2P}{\tau} t & (0 < t < \dfrac{\tau}{2}) \\ F_2 = m\dfrac{dv}{dt} = \dfrac{2P}{\tau}(\tau - t) & (\dfrac{\tau}{2} < t < \tau) \end{cases}$$

两边积分:

$$\therefore \begin{cases} \int_0^{v_1} m\,dv = \dfrac{2P}{\tau}\int_0^{\frac{\tau}{2}} t\,dt & (0 < t < \dfrac{\tau}{2}) \\ \int_{v_1}^{v_2} m\,dv = \dfrac{2P}{\tau}\int_{\frac{\tau}{2}}^{\tau} (\tau - t)\,dt & (\dfrac{\tau}{2} < t < \tau) \end{cases}$$

$$\Rightarrow v_1 = \frac{P}{m\tau}\left(\frac{\tau}{2}\right)^2 = \frac{P}{4m}\tau \qquad \Rightarrow v_2 = \frac{P\tau}{2m}$$

$$\therefore \Delta E_k = \frac{1}{2} m v_2^2$$

$$= \frac{1}{2} m \left(\frac{P\tau}{2m}\right)^2 = \frac{1}{8}\cdot\frac{(P\tau)^2}{m}$$