Review:

动力学普遍方程: $$\sum_{i=1}^{n}(\vec{F}_i - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0$$

or $$\sum_{i=1}^{3n}(F_i - m_i\ddot{x}_i)\cdot\delta x_i = 0$$

基本形式拉格朗日方程: $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

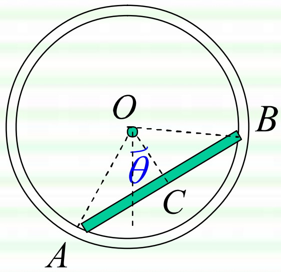
$$Q_\alpha = -\frac{\partial V}{\partial q_\alpha}$$

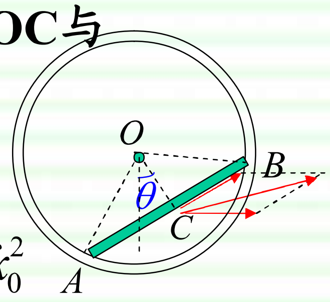
保守系的拉格朗日方程: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

本次课:小振动和冲击运动

上次例题:

例题2、图中放在一直线轨道上的圆环O的质量为m_2，半径为r。圆环与轨道间有足够的摩擦力阻止圆环滑动。圆环内有一质量为m_1，长为l的匀质杆AB，其中点C与圆环的中心O的距离为$r/2^{1/2}$,则杆长$l=2^{1/2}r$。杆与圆环间摩擦不计。写出系统的运动微分方程。





解: $i=2$，取圆环中心O的坐标x_0以及OC与垂直线的交角θ为广义坐标。

圆环作纯滚动，则: $\omega=\frac{\dot{x}_0}{r}$

圆环动能: $T_2=\frac{1}{2}m_2\dot{x}_0^2+\frac{1}{2}(m_2r^2)\omega^2=m_2\dot{x}_0^2$

杆AB动能:

$$T_1=\frac{1}{2}m_1(\dot{x}_0^2+\frac{1}{2}r^2\dot{\theta}^2+\sqrt{2}r\dot{x}_0\dot{\theta}\cos\theta)+\frac{1}{2}\cdot\frac{1}{12}m_1(\sqrt{2}r)^2\dot{\theta}^2]$$

总动能:

$$T=T_1+T_2=(m_2+\frac{m_1}{2})\dot{x}_0^2+\frac{1}{3}m_1r^2\dot{\theta}^2+\frac{\sqrt{2}}{2}m_1r\dot{x}_0\dot{\theta}\cos\theta$$

零势面：O点的水平面: $V=-\frac{\sqrt{2}}{2}m_1rg\cos\theta$

$$\therefore L = T - V$$
$$= (m_2 + \frac{m_1}{2})\dot{x}_0^2 + \frac{1}{3}m_1 r^2\dot{\theta}^2 + \frac{\sqrt{2}}{2}m_1 r\dot{x}_0\dot{\theta}\cos\theta + \frac{\sqrt{2}}{2}m_1 rg\cos\theta$$

代入拉格朗日方程中:

$$x_0: \quad (2m_2 + m_1)\ddot{x}_0 + \frac{\sqrt{2}}{2}m_1 r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0$$

$$\theta: \quad 4r\ddot{\theta} + 3\sqrt{2}\ddot{x}_0\cos\theta + 3\sqrt{2}g\sin\theta = 0$$

只考虑微振动则: $(2m_2 + m_1)\ddot{x}_0 + \frac{\sqrt{2}}{2}m_1 r\ddot{\theta} = 0$

$$4r\ddot{\theta} + 3\sqrt{2}\ddot{x}_0 + 3\sqrt{2}g\theta = 0$$

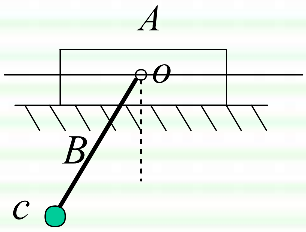
消去$\ddot{x}_0$得: $\frac{8m_2 + m_1}{2m_2 + m_1}r\ddot{\theta} + 3\sqrt{2}g\theta = 0$

$$\Rightarrow \ddot{\theta} + 3\sqrt{2}\frac{g}{r}\frac{2m_2 + m_1}{8m_2 + m_1}\theta = 0$$

($\omega^2 = 3\sqrt{2}\frac{g}{r}\frac{2m_2 + m_1}{8m_2 + m_1}$)

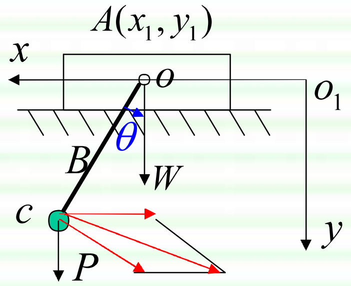
此乃小振动问题!

例题3、如图示，物块A沿水平面滑动，细杆B一端用铰链与A相连，另一端与球C相固连所组成椭圆摆。设细杆B长为l，A和C的质量分别为m_1和m_2，细杆B重量不计，求：椭圆摆运动微分方程。



解：1）A、C组成系统，$i=2$

平面坐标系O_1xy，广义坐标θ和x_1



2) 先找A和C(x_2,y_2)的坐标:

$$\begin{cases} x_2 = x_1 + l\sin\theta \\ y_2 = l\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = \dot{x}_1 + l\dot{\theta}\cos\theta \\ \dot{y}_2 = -l\dot{\theta}\sin\theta \end{cases}$$

A: $T_A = \frac{1}{2}m_1\dot{x}_1^2$

C: $v_c = (\dot{x}_2^2 + \dot{y}_2^2)^{1/2} = (\dot{x}_1^2 + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta)^{1/2}$

Or: $|\vec{v}_c|^2 = |\dot{x}_1\vec{i} + l\dot{\theta}\vec{e}_{\perp oc}|^2 = \dot{x}_1^2 + l^2\dot{\theta}^2 - 2\dot{x}_1 l\dot{\theta}\cos(\pi - \theta)$

$$\therefore T_c = \frac{1}{2}m_2 v_c^2 = \frac{1}{2}m_2(\dot{x}_1^2 + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta)$$

总动能：$T = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 + m_2 l\dot{x}_1\dot{\theta}\cos\theta$

势能：　$V = -m_2 gl\cos\theta$

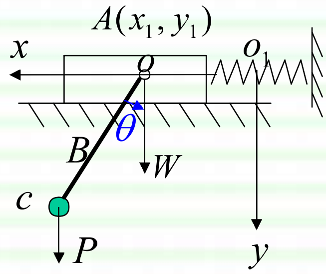
$$\therefore L = T - V$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 + m_2 l\dot{x}_1\dot{\theta}\cos\theta + m_2 gl\cos\theta$$

代入拉格朗日方程中：　$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0$　***i*=2**

考虑微小摆动：

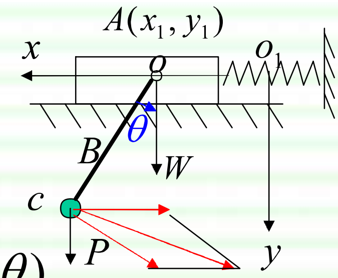
$$\ddot{\theta} + \frac{m_1 + m_2}{m_1}\cdot\frac{g}{l}\theta = 0$$



若**A**用弹簧连接在固定点处，?

此时系统势能将附加弹性势能项：　$\frac{1}{2}kx_1^2$

解：i=2　　广义坐标θ和x_1



$T_A = \frac{1}{2} m_1 \dot{x}_1^2$　　$\vec{v}_c = \dot{x}_1 \vec{i} + l\dot{\theta}\vec{e}_{cp}$

$\therefore v_c^2 = \dot{x}_1^2 + l^2\dot{\theta}^2 - 2\dot{x}_1 l\dot{\theta}\cos(\pi - \theta)$

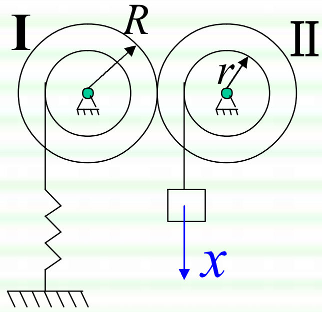
$\therefore T_c = \frac{1}{2} m_2 v_c^2 = \frac{1}{2} m_2 (\dot{x}_1^2 + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta)$

总动能：$T = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_2 l \dot{x}_1 \dot{\theta}\cos\theta$

势能：$V = -m_2 g l \cos\theta + \frac{1}{2} k x_1^2$　　$\therefore L = T - V$

代入拉格朗日方程中：$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

例题4、两个相同的塔轮；半径为r的鼓轮上饶有细绳，轮I绕有铅直弹簧(k)，轮II挂一重物。塔轮对轴的转动惯量均为J，重物质量为m。求此振动系统的固有频率。



解：$i=1$，取$q=x$，$\dot{x}=\omega r$

系统动能：$T=\frac{1}{2}m\dot{x}^2+2\times\frac{1}{2}J(\frac{\dot{x}}{r})^2$

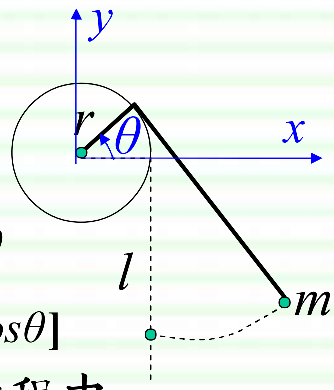
系统势能：$V=\frac{1}{2}kx^2-mgx$

$$\therefore L=T-V=\frac{1}{2}m\dot{x}^2+J(\frac{\dot{x}}{r})^2-\frac{1}{2}kx^2$$

代入保守系的拉格朗日方程中：

$$\Rightarrow (m+\frac{2J}{r^2})\ddot{x}+kx=0 \qquad \Rightarrow \omega_0=\sqrt{\frac{kr^2}{2J+mr^2}}$$

例题5、质量为m的质点悬挂在轻绳一端，绳的另一端绕在半径为r的固定圆柱体上，构成一个摆。设平衡时绳的下垂部分长为l，试求摆的运动微分方程。



解：i=1，取 $q=\theta$

系统动能：$T=\frac{1}{2}m[(l+r\theta)\dot{\theta}]^2$

$$\begin{cases} x=r\cos\theta+(l+r\theta)\sin\theta \\ y=r\sin\theta-(l+r\theta)\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x}=\dot{\theta}(l+r\theta)\cos\theta \\ \dot{y}=\dot{\theta}(l+r\theta)\sin\theta \end{cases}$$

系统势能：$V=mg[(l+r\sin\theta)-(l+r\theta)\cos\theta]$

$\therefore L=T-V$ 代入保守系的拉格朗日方程中：

$$(l+r\theta)\ddot{\theta}+r\dot{\theta}^2+g\sin\theta=0$$

（$\dot{\theta}^2$ → 0，$\sin\theta$ → θ）

$$\Rightarrow \ddot{\theta}+\frac{g}{l}\theta=0$$

冲击运动的拉格朗日方程

冲击运动的特点:

1、冲力大；作用时间短；

2、可忽略冲击造成的位形变化；

3、$\lim_{\Delta t \to 0} \int_t^{t+\Delta t} \vec{F} dt$ 是有限的。

一、矢量力学关于质点冲击运动的动力学方程:

$$\because\ m\frac{d\vec{v}}{dt}=\vec{F}$$

$$\therefore\ \lim_{\Delta t\to 0}\int_t^{t+\Delta t} m\frac{d\vec{v}}{dt}\cdot dt=\lim_{\Delta t\to 0}\int_t^{t+\Delta t}\vec{F}dt$$

$$m\left[\lim_{\Delta t\to 0}\vec{V}\left(t+\Delta t\right)-\vec{V}\left(t\right)\right]=\vec{I}$$

$$\Rightarrow \Delta\vec{P}=\vec{I}$$

二、分析力学关于系统的冲击运动的动力学方程:

$$\Rightarrow \Delta P_k=I_k$$

$$\because\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k\left(k=1,2\cdots s\right)$$

以 **d*t*** 乘上式且对碰撞时间 ***Δt*** 积分:

$$\therefore \int_t^{t+\Delta t} d\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \int_t^{t+\Delta t} \frac{\partial T}{\partial q_k} dt = \int_t^{t+\Delta t} Q_k dt$$

?? 　　　0　　　?

$\because \Delta t \to 0$

$$? = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} \left(\sum_{i=1}^{3n} F_i \frac{\partial x_i}{\partial q_k}\right) dt = \sum_{i=1}^{3n}\left[\lim_{\Delta t \to 0}\left(\int_t^{t+\Delta t} F_i dt\right)\frac{\partial x_i}{\partial q_k}\right]$$

$$= \sum_{i=1}^{3n} I_i \frac{\partial x_i}{\partial q_k} = I_k$$　广义冲量

$$?? = \int_t^{t+\Delta t} dP_k = P_{k2} - P_{k1}$$

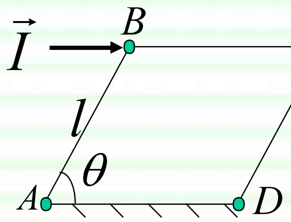
$$\therefore P_{k2} - P_{k1} = I_k (k = 1, 2, \ldots, s)$$

即：$$\left[\frac{\partial T}{\partial \dot{q}_k}\right]_{t_2} - \left[\frac{\partial T}{\partial \dot{q}_k}\right]_{t_1} = I_k \left(k = 1, 2, \cdots s\right)$$

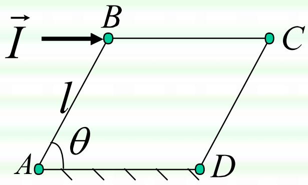
一般取零

关于系统冲击运动的拉格朗日方程。

例题1: 铰链平行四边形ABCD置于光滑水平面上， AB，CD杆的质量均为m_1， BC杆质量为m_2， AD杆固定不动， ∠BAD=θ。现于B点处作用一冲量I，如图示。求B点速度及整个系统所获得的动能。



解：i=1。 取$q=\theta$



杆AB和杆BC的质心速度分别为：

$$u_1 = \frac{l}{2}\dot{\theta}, \quad u_2 = l\dot{\theta}$$

$$I_A = \frac{1}{3}m_1 l^2;$$

系统的动能为：

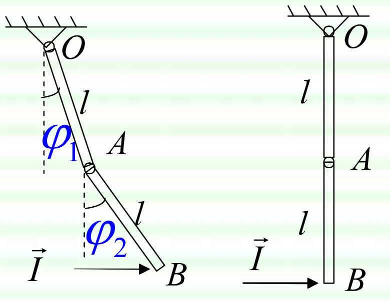
$$T = 2 \times \frac{1}{2} I_A \dot{\theta}^2 + \frac{1}{2} m_2 u_2^{\,2} = (\frac{m_1}{3} + \frac{m_2}{2}) l^2 \dot{\theta}^2$$

$\because dx_B = -l d\theta \sin\theta$ 广义冲量：$I_k = I \dfrac{\partial x_B}{\partial \theta} = -Il\sin\theta$

代入 $\left(\dfrac{\partial T}{\partial \dot{\theta}}\right)_2 - \left(\dfrac{\partial T}{\partial \dot{\theta}}\right)_1 = I_k$ 中：

$$\Rightarrow u_2 = \frac{-3I\sin\theta}{2m_1 + 3m_2} \qquad T = \frac{3I^2\sin^2\theta}{2(2m_1 + 3m_2)}$$

例题2： 两个长均为l，质量均为m的均质杆在A处铰链后悬挂在O轴上，并在B端受到冲量I的作用，求碰后两杆的角速度。



解： $i=2$。 取$q_1=\varphi_1$和$q_2=\varphi_2$

$$\left(\frac{\partial T}{\partial \dot{q}_k}\right)_2-\left(\frac{\partial T}{\partial \dot{q}_k}\right)_1=I_k$$

两杆质心速度分别为：

$$u_1=\frac{1}{2}l\dot{\varphi}_1, \qquad u_2=l\dot{\varphi}_1+\frac{l}{2}\dot{\varphi}_2$$

$$\therefore T=\frac{1}{2}I_0\dot{\varphi}_1^{\ 2}+\frac{1}{2}mu_2^{\ 2}+\frac{1}{2}I_{C_2}\dot{\varphi}_2^{\ 2}$$

其中： $I_0=\frac{1}{3}ml^2$; $\qquad I_{C_2}=\frac{1}{12}ml^2$

$$\therefore T=\frac{1}{2}ml^2\left(\frac{4}{3}\dot{\varphi}_1^{\ 2}+\dot{\varphi}_1\dot{\varphi}_2+\frac{1}{3}\dot{\varphi}_2^{\ 2}\right)$$

$$T=\frac{1}{2}ml^2\left(\frac{4}{3}\dot{\varphi}_1{}^2+\dot{\varphi}_1\dot{\varphi}_2+\frac{1}{3}\dot{\varphi}_2{}^2\right)$$

现求广义冲量: $\boldsymbol{I}_k=\sum_{i=1}^{n}\vec{\boldsymbol{I}}_i\cdot\frac{\partial\vec{\boldsymbol{r}}_i}{\partial\boldsymbol{q}_k}=\vec{\boldsymbol{I}}\cdot\frac{\partial\vec{\boldsymbol{r}}_B}{\partial\boldsymbol{q}_k}=\boldsymbol{I}\frac{\partial\boldsymbol{x}_B}{\partial\boldsymbol{q}_k}$

$$\because\ x_B=l\sin\varphi_1+l\sin\varphi_2$$

$\therefore\ I_{\varphi_1}=I\frac{\partial x_B}{\partial\varphi_1}=Il\cos\varphi_1\xrightarrow{\varphi_1\to 0}Il$ 同理: $I_{\varphi_2}=Il$

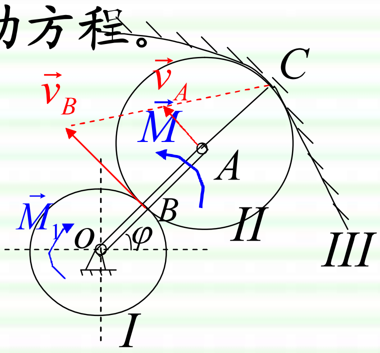
$$\Rightarrow\dot{\varphi}_1=-\frac{6I}{7ml};\qquad\dot{\varphi}_2=\frac{30I}{7ml}$$

? 若I作用点在AB杆质心处，情况又如何？

I'_{φ_1} 不变

$I'_{\varphi_2}=I\frac{\frac{l}{2}\delta\varphi_2}{\delta\varphi_2}=\frac{Il}{2}$ $\Rightarrow\dot{\varphi}_1=\frac{3I}{7ml};\qquad\dot{\varphi}_2=\frac{6I}{7ml}$

小论文： 图示水平面内行星轮系中，长为l的曲柄OA绕O轴转动，其端点A由铰链连接一r_2的齿轮II，已知$r_2=1.5r_1$，曲柄OA质量为m，齿轮I、II的质量分别为m_1、m_2。在曲柄上作用一力偶M，齿轮I上有阻力偶M_1，求曲柄运动方程。



解： $i=1$，选曲柄转角φ为广义坐标，轮心A也是曲柄端点A。

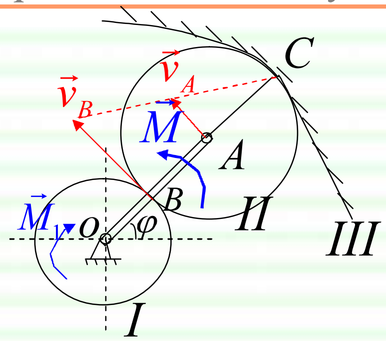
$$\therefore v_A=(r_1+r_2)\omega=l\dot{\varphi} \qquad v_C=0$$

而：$v_C=v_A-r_2\dot{\varphi}_{II}$

则轮II的角速度：$\dot{\varphi}_{II}=\dfrac{l\dot{\varphi}}{r_2}=\dfrac{5}{3}\dot{\varphi}$

又轮I与轮II的啮合处B的速度：$v_B=2v_A=2l\dot{\varphi}$

则轮I的角速度：$\dot{\varphi}_I=\dfrac{v_B}{r_1}=\dfrac{2l\dot{\varphi}}{r_1}=5\dot{\varphi}$



曲柄的动能:

$$T_{qb}=\frac{1}{2}J_{qbo}\dot{\varphi}^2=\frac{1}{2}\cdot\frac{1}{3}ml^2\dot{\varphi}^2=\frac{1}{6}ml^2\dot{\varphi}^2$$

齿轮I的动能:

$$T_I=\frac{1}{2}J_{Io}\dot{\varphi}_I{}^2=\frac{1}{2}\cdot\frac{1}{2}m_1r_1{}^2\dot{\varphi}_I{}^2=m_1l^2\dot{\varphi}^2$$

齿轮II的动能:

$$T_{II}=\frac{1}{2}m_2v_A^2+\frac{1}{2}J_{IIA}\dot{\varphi}_{II}{}^2$$

$$=\frac{1}{2}m_2(l\dot{\varphi})^2+\frac{1}{2}\cdot\frac{1}{2}m_2r_2^2(\frac{l\dot{\varphi}}{r_2})^2=\frac{3}{4}m_2l^2\dot{\varphi}^2$$

系统的总动能:

$$T=(\frac{1}{6}m+m_1+\frac{3}{4}m_2)l^2\dot{\varphi}^2$$

计算广义力:

给曲柄一虚位移δφ，则 $\delta\varphi_I = 5\delta\varphi$ Q_φ

则主动力的虚功为:

$$\delta W = M\delta\varphi - M_1\delta\varphi_I = M\delta\varphi - 5M_1\delta\varphi = (M-5M_1)\delta\varphi$$

代入拉格朗日方程中得:

$$(\frac{1}{3}m + 2m_1 + \frac{3}{2}m_2)l^2\ddot{\varphi} = M - 5M_1$$

$$\Rightarrow \ddot{\varphi} = \frac{M-5M_1}{(\frac{1}{3}m + 2m_1 + \frac{3}{2}m_2)l^2}$$

设t=0时，$\varphi = \varphi_0;\quad \dot{\varphi} = \dot{\varphi}_0$

$$\Rightarrow \varphi = \frac{M-5M_1}{(\frac{1}{3}m + 2m_1 + \frac{3}{2}m_2)l^2}t^2 + \dot{\varphi}_0 t + \varphi_0$$

作业：一、习题5.11 、 5.12、5.15

下次课:初积分问题