Review:

①动力学普遍方程:

$$\sum_{i=1}^{3n}\left(F_i - m_i\ddot{x}_i\right)\cdot\delta x_i = 0$$

②广义力:

$$Q_k = \sum_{i=1}^{3n} F_i \frac{\partial x_i}{\partial q_k} \qquad \left(k = 1, 2, \cdots s\right)$$

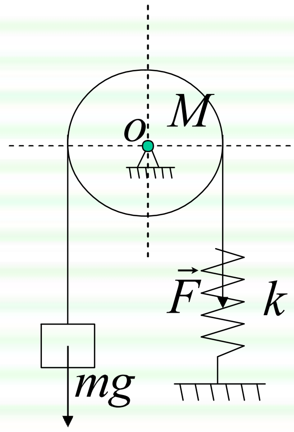
③广义惯性力:

$$P_k = \sum_{i=1}^{3n}\left(m_i\ddot{x}_i \frac{\partial x_i}{\partial q_k}\right) \qquad (k=1, 2\ldots s)$$

④基本形式的拉格朗日方程:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \qquad (k = 1, 2, \ldots s)$$

上次作业：求重物振动周期。



解：i=1，选滑轮转角φ为广义坐标

$$T=\frac{1}{2}MR^2\cdot\dot{\varphi}^2+\frac{1}{2}m(R\dot{\varphi})^2=\frac{1}{2}(M+m)R^2\dot{\varphi}^2$$

φ=0时，系统平衡，$mg=k\delta$

主动力：mg、$k(R\varphi+\delta)$

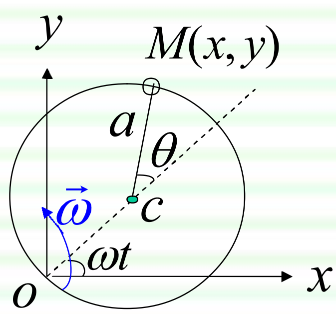
广义力Q_k：
$$\delta W=mgR\delta\varphi-k(\delta+R\varphi)R\delta\varphi$$
$$=(mg-k\delta-kR\varphi)R\delta\varphi=-kR^2\varphi\delta\varphi$$

Q_k

代入方程：
$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k$$

$$\Rightarrow\ddot{\varphi}+\frac{k}{M+m}\varphi=0 \qquad \Rightarrow T=?$$

作业5.6解小环i=1，$q=\theta$



OC以角速度ω绕过O点的竖直轴转动。

$$\because \begin{cases} x = a\cos\omega t + a\cos(\omega t+\theta) \\ y = a\sin\omega t + a\sin(\omega t+\theta) \end{cases}$$

$$\therefore T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) = \frac{1}{2}m\left[a^2\omega^2 + a^2\left(\omega+\dot{\theta}\right)^2 + 2a^2\omega\left(\omega+\dot{\theta}\right)\cos\theta\right]$$

$$= \frac{1}{2}m\left[4a^2\omega^2\cos^2\frac{\theta}{2} + 4a^2\omega\dot{\theta}\cos^2\frac{\theta}{2} + a^2\dot{\theta}^2\right]$$

水平向不受主动力 $Q_\theta = 0$ 而 $Q_\theta = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\theta}\right) - \frac{\partial T}{\partial q_\theta}$

$$\frac{d}{dt}\left\{\frac{1}{2}m\left[2a^2\left(\omega+\dot{\theta}\right) + 2a^2\omega\cos\theta\right]\right\} - \frac{1}{2}m2a^2\omega\left(\omega+\dot{\theta}\right)\left(-\sin\theta\right) = 0$$

$$\Rightarrow \ddot{\theta} + \omega^2\sin\theta = 0$$

分析:(一): 小环$i=1$， θ为广义坐标，

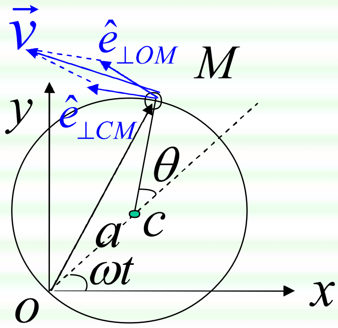
$$\therefore T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)$$

求法(二): $\because \vec{v}=\overline{oM}\omega\vec{e}_{\perp OM}+a\dot{\theta}\vec{e}_{\perp CM}$

$$\overline{oM}=2a\cos\frac{\theta}{2}$$

$$\therefore v^2=(\overline{oM}\omega)^2+(a\dot{\theta})^2-2(a\dot{\theta})(\overline{oM}\omega)\cos(\pi-\frac{\theta}{2})$$

$$=4a^2\omega^2\cos^2\frac{\theta}{2}+4a^2\omega\dot{\theta}\cos^2\frac{\theta}{2}+a^2\dot{\theta}^2$$



结果一样

求法(三): 也可采用动坐标系,结果将一致

第16讲 保守系的拉格朗日方程

保守力？ 有势力？

$V(\vec{r})$ $U(\vec{r},t)$ 两者的关系？

势能？ 势函数？

故：保守系中必存在着势能V，是坐标的函数：

$$\vec{F}_i = -\nabla V_i \qquad (i = 1, 2, \cdots, n)$$

$$or \qquad F_{ix} = -\frac{\partial V_i}{\partial x_i}, F_{iy} = -\frac{\partial V_i}{\partial y_i}, F_{iz} = -\frac{\partial V_i}{\partial z_i}$$

注意：

势函数和势能都需以广义坐标表示。

$$\therefore Q_k = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \sum_{i=1}^{n} (F_{ix} \cdot \frac{\partial x_i}{\partial q_k} + F_{iy} \cdot \frac{\partial y_i}{\partial q_k} + F_{iz} \cdot \frac{\partial z_i}{\partial q_k})$$

$$= \sum_{i=1} (-\frac{\partial V_i}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_k} - \frac{\partial V_i}{\partial y_i} \cdot \frac{\partial y_i}{\partial q_k} - \frac{\partial V_i}{\partial z_i} \cdot \frac{\partial z_i}{\partial q_k})$$

$$= -\sum_{i=1} \frac{\partial V_i}{\partial q_k} = -\frac{\partial V}{\partial q_k} (k = 1, 2, ..., s)$$

基本形式的拉格朗日方程可改写为：$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_k}) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k}$

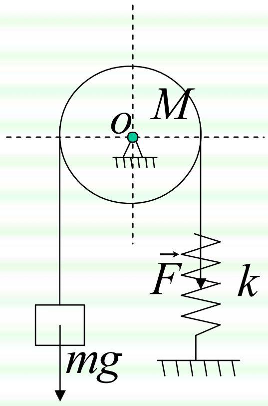
势能V中一般不包含广义速度$\dot{q}_k$： $\Rightarrow \frac{\partial V}{\partial \dot{q}_k} = 0$

令：$L = T - V$

$$\Rightarrow \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) - \frac{\partial L}{\partial q_k} = 0$$ 保守系的拉格朗日方程

L的物理意义？ 称 L 为拉格朗日函数

作业另解:



解:系统i=1,选滑轮转角φ为广义坐标

动能: $T=\frac{1}{2}(M+m)R^2\dot{\varphi}^2$

势能: φ=0时,系统平衡,$mg=k\delta$

$$\therefore V=-mgR\varphi+[\frac{1}{2}k(\delta+R\varphi)^2-\frac{1}{2}k\delta^2]=\frac{1}{2}kR^2\varphi^2$$

$$\therefore L=T-V=\frac{1}{2}(M+m)R^2\dot{\varphi}^2-\frac{1}{2}kR^2\varphi^2$$

相对性

代入拉氏方程: $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k})-\frac{\partial L}{\partial q_k}=0$

$$\Rightarrow \ddot{\varphi}+\frac{k}{M+m}\varphi=0 \qquad \Rightarrow T=?$$

例1、一质量为m的珠子在光滑的金属丝上滑动。用柱坐标，金属丝的方程是$\rho=a$，$z=b\varphi$。重力沿z轴正方向。珠子自$\rho=a$，$z=0$，$\varphi=0$处静止释放，求：①珠子的运动方程$\varphi(t)$和$z(t)$;

**②金属丝对珠子的作用力。

解：①珠子的动能：$T=\frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\varphi}^2+\frac{1}{2}m\dot{z}^2$

约束方程：$\rho=a,\quad z=b\varphi$

***i*=1**，选 z 为广义坐标，有：$T=\frac{1}{2}m\frac{a^2+b^2}{b^2}\dot{z}^2$

珠子的势能：$V=-mgz \qquad \therefore L=T-V$

代入***L's***方程式中：$m\frac{a^2+b^2}{b^2}\ddot{z}-mg=0$

积分上式且利用初始条件：**t=0**时，$z=0,\dot{z}=0$

可得：
$$\begin{cases} z=\dfrac{b^2}{2\left(a^2+b^2\right)}gt^2 \\ \varphi=\dfrac{z}{b}=\dfrac{b}{2\left(a^2+b^2\right)}gt^2 \end{cases}$$
为珠子的运动方程

**②金属丝对珠子的作用求法可有多种办法:

方法㈠: 分析力学方法: 解除约束，变为主动力。

此时 $i=3$，选 ρ,φ,z为广义坐标。

广义力为: $Q_\rho, \rho Q_\varphi, Q_z$

$$T=\frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\varphi}^2+\frac{1}{2}m\dot{z}^2$$

$$\therefore L=T-V=\frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\varphi}^2+\frac{1}{2}m\dot{z}^2+mgz$$

由式:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right]-\frac{\partial L}{\partial q_k}=Q_k$$

得:

$$\begin{cases} m\ddot{\rho}-m\rho\dot{\varphi}^2=Q_\rho \\ m\rho^2\ddot{\varphi}=Q_\varphi\rho \\ m\ddot{z}-mg=Q_z \end{cases}$$

将 $\rho=a$，及前面解得的$\varphi(t)$和$z(t)$代入，可得。

$$\Rightarrow Q_\rho=\frac{-mab^2g^2t^2}{\left(a^2+b^2\right)^2};\quad Q_\varphi=\frac{mabg}{a^2+b^2};\quad Q_z=\frac{-ma^2g}{a^2+b^2}$$

方法(二): 用矢量力学方法:

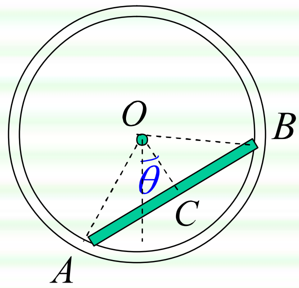
$$\because \vec{a}=(\ddot{\rho}-\rho\dot{\varphi}^2)\ \vec{e}_\rho+\left(\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi}\right)\vec{e}_\varphi+\ddot{z}\vec{e}_z$$

$$\therefore\begin{cases} m\left(\ddot{\rho}-\rho\dot{\varphi}^2\right)=Q_\rho \\ m\left(\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi}\right)=Q_\varphi \\ m\ddot{z}=mg+Q_z \end{cases}$$

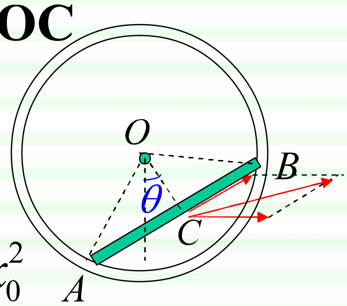
考虑到: $\begin{cases}\rho=a\\ \dot{\rho}=0\end{cases}$

得完全相同的结果。

例题2、图中放在一直线轨道上的圆环O的质量为m_2，半径为r。圆环与轨道间有足够的摩擦力阻止圆环滑动。圆环内有一质量为m_1，长为l的匀质杆AB，其中点C与圆环的中心O的距离为$r/2^{1/2}$,则杆长$l=2^{1/2}r$。杆与圆环间摩擦不计。写出系统的运动微分方程。



解： i=2，取圆环中心O的坐标x_0以及OC与垂直线的交角θ为广义坐标。



圆环作纯滚动，则： $\omega = \dfrac{\dot{x}_0}{r}$

圆环动能： $T_2 = \dfrac{1}{2}m_2\dot{x}_0^2 + \dfrac{1}{2}(m_2 r^2)\omega^2 = m_2\dot{x}_0^2$

杆AB动能：

$$T_1 = \frac{1}{2}m_1(\dot{x}_0^2 + \frac{1}{2}r^2\dot{\theta}^2 + \sqrt{2}r\dot{x}_0\dot{\theta}\cos\theta) + \frac{1}{2}\cdot\frac{1}{12}m_1(\sqrt{2}r)^2\dot{\theta}^2]$$

总动能：

$$T = T_1 + T_2 = (m_2 + \frac{m_1}{2})\dot{x}_0^2 + \frac{1}{3}m_1 r^2\dot{\theta}^2 + \frac{\sqrt{2}}{2}m_1 r\dot{x}_0\dot{\theta}\cos\theta$$

零势面：O点的水平面： $V = -\dfrac{\sqrt{2}}{2}m_1 rg\cos\theta$

$$\therefore L = T - V$$
$$= (m_2 + \frac{m_1}{2})\dot{x}_0^2 + \frac{1}{3}m_1 r^2\dot{\theta}^2 + \frac{\sqrt{2}}{2}m_1 r\dot{x}_0\dot{\theta}\cos\theta + \frac{\sqrt{2}}{2}m_1 rg\cos\theta$$

代入拉格朗日方程中:

$$x_0: \quad (2m_2 + m_1)\ddot{x}_0 + \frac{\sqrt{2}}{2}m_1 r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0$$

$$\theta: \quad 4r\ddot{\theta} + 3\sqrt{2}\ddot{x}_0\cos\theta + 3\sqrt{2}g\sin\theta = 0$$

只考虑微振动则: $(2m_2 + m_1)\ddot{x}_0 + \frac{\sqrt{2}}{2}m_1 r\ddot{\theta} = 0$

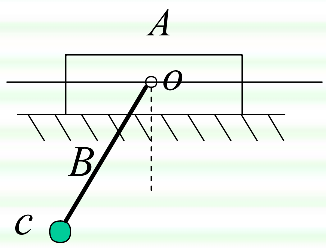
$$4r\ddot{\theta} + 3\sqrt{2}\ddot{x}_0 + 3\sqrt{2}g\theta = 0$$

消去$\ddot{x}_0$得: $\frac{8m_2 + m_1}{2m_2 + m_1}r\ddot{\theta} + 3\sqrt{2}g\theta = 0$

$$\Rightarrow \ddot{\theta} + 3\sqrt{2}\frac{g}{r}\frac{2m_2 + m_1}{8m_2 + m_1}\theta = 0$$

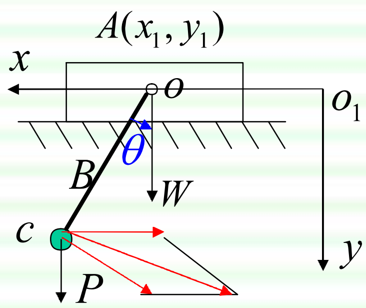
ω^2

例题3、如图示，物块A沿水平面滑动，细杆B一端用铰链与A相连，另一端与球C相固连所组成椭圆摆。设细杆B长为l，A和C的质量分别为m_1和m_2，细杆B重量不计，**求：**椭圆摆运动微分方程。



解：1）A、C组成系统，$i=2$

平面坐标系O_1xy，广义坐标θ和x_1



2) 先找A和C(x_2,y_2)的坐标：

$$\begin{cases} x_2 = x_1 + l\sin\theta \\ y_2 = l\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = \dot{x}_1 + l\dot{\theta}\cos\theta \\ \dot{y}_2 = -l\dot{\theta}\sin\theta \end{cases}$$

A: $T_A = \dfrac{1}{2} m_1 \dot{x}_1^2$

C: $v_c = (\dot{x}_2^2 + \dot{y}_2^2)^{1/2} = (\dot{x}_1^2 + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta)^{1/2}$

Or: $\vec{v}_c = \dot{x}_1\vec{i} + l\dot{\theta}\vec{e}_{cp} = \dot{x}_1^2 + l^2\dot{\theta}^2 - 2\dot{x}_1 l\dot{\theta}\cos(\pi - \theta)$

$$\therefore T_c = \frac{1}{2} m_2 v_c^2 = \frac{1}{2} m_2 (\dot{x}_1^2 + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta)$$

总动能：$T = \dfrac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \dfrac{1}{2} m_2 l^2\dot{\theta}^2 + m_2 l\dot{x}_1\dot{\theta}\cos\theta$

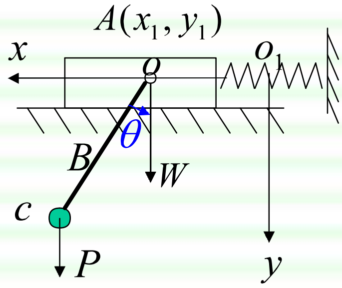
势能： $V=-m_2 gl\cos\theta$

$$\therefore L=T-V$$
$$=\frac{1}{2}(m_1+m_2)\dot{x}_1^2+\frac{1}{2}m_2 l^2\dot{\theta}^2+m_2 l\dot{x}_1\dot{\theta}\cos\theta+m_2 gl\cos\theta$$

代入拉格朗日方程中： $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})-\frac{\partial L}{\partial q}=0$ ***i*=2**

考虑微小摆动：

$$\ddot{\theta}+\frac{m_1+m_2}{m_1}\cdot\frac{g}{l}\theta=0$$



若**A**用弹簧连接在固定点处，?

此时系统势能将附加弹性势能项： $\frac{1}{2}kx_1^2$

作业：习题5.5，5.7，5.8

下次课:小振动和冲击运动