Review: 分析力学的拉格朗日方程

达朗贝尔原理: $\sum_{i=1}^{n}[\vec{F}_i + \vec{R}_i + (-m_i\ddot{\vec{r}}_i)] = 0$

达朗贝尔-拉格朗日方程:
又称动力学普遍方程: $\sum_{i=1}^{n}(\vec{F}_i - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0$

基本形式拉格朗日方程: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$

$Q_\alpha = -\frac{\partial V}{\partial q_\alpha}$

保守系的拉格朗日方程: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$

初积分: 只要 L 不显含 t 或某个广义坐标，就有广义能量积分和相应的广义动量积分。

第23讲 分析力学的哈密顿正则方程

1、拉格朗日函数L是广义坐标的函数:

$$L = L(q_\alpha, \dot{q}_\alpha, t) \qquad (\alpha = 1, 2, \cdots s)$$

2、拉格朗日方程是二阶常微分方程组。

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \quad (\alpha = 1, 2, \cdots s)$$

若把广义速度 $\dot{q}_\alpha$ 变换为广义动量 $P_\alpha = \dfrac{\partial L}{\partial \dot{q}_\alpha}$ ①

1、使方程组降阶变为一阶常微分方程组。

$$\frac{dp_\alpha}{dt} - \frac{\partial L}{\partial q_\alpha} = 0 \quad (\alpha = 1, 2, \cdots s) \quad or \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$ ②

2、则函数将变为广义坐标和广义动量的函数:

$$\overline{L} = \overline{L}(q_\alpha, p_\alpha, t) \qquad (\alpha = 1, 2, \cdots s)$$

如何变换?

一、勒襄特变换:

由一组独立变数变换为另一组独立的变数，在数学上叫勒襄特变换。

勒襄特变换的基本法则:

新的函数（如**g**或**H**）等于不要的变量（如x或q'_α）乘以原来的函数对该变量的偏微商

$$\left(u = \frac{\partial f}{\partial x} \quad or \quad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \qquad \alpha = 1, 2.., s \right)$$

再减去原来的函数（如f或**L**）。即:

$$g = \left(\frac{\partial f}{\partial x} x - f \right) \qquad or \qquad H = \sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha - L$$

二、正则方程:

$$\because dL=\sum_{\alpha=1}^{s}\left(\frac{\partial L}{\partial \dot{q}_\alpha}d\dot{q}_\alpha+\frac{\partial L}{\partial q_\alpha}dq_\alpha\right)+\frac{\partial L}{\partial t}dt$$

$$H=\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha-L=\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L$$

$$\because p_\alpha=\frac{\partial T}{\partial \dot{q}_\alpha}=\frac{\partial L}{\partial \dot{q}_\alpha}$$

由保守系拉格朗日方程知: $\dot{p}_\alpha=\frac{\partial L}{\partial q_\alpha}$

$$\therefore dL=\sum_{\alpha=1}^{s}\left(p_\alpha d\dot{q}_\alpha+\dot{p}_\alpha dq_\alpha\right)+\frac{\partial L}{\partial t}dt$$

$$=\sum_{\alpha=1}^{s}[d(p_\alpha\dot{q}_\alpha)-\dot{q}_\alpha dp_\alpha]+\sum_{\alpha=1}^{s}\dot{p}_\alpha dq_\alpha+\frac{\partial L}{\partial t}dt$$

$$\therefore d[\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L]=\sum_{\alpha=1}^{s}\dot{q}_\alpha dp_\alpha-\sum_{\alpha=1}^{s}\dot{p}_\alpha dq_\alpha-\frac{\partial L}{\partial t}dt$$

H

即: $dH=\sum_{\alpha=1}^{s}\dot{q}_\alpha dp_\alpha-\sum_{\alpha=1}^{s}\dot{p}_\alpha dq_\alpha-\frac{\partial L}{\partial t}dt$

$$dH = \sum_{\alpha=1}^{s} \dot{q}_\alpha dp_\alpha - \sum_{\alpha=1}^{s} \dot{p}_\alpha dq_\alpha - \frac{\partial L}{\partial t} dt \qquad ①$$

$$H(p,q,t) = \sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha - L$$

***H*的全微分为：**

$$dH = \sum_{\alpha=1}^{s} \left(\frac{\partial H}{\partial q_\alpha} dq_\alpha + \frac{\partial H}{\partial p_\alpha} dp_\alpha \right) + \frac{\partial H}{\partial t} dt \qquad ②$$

比较两式有：

$$\begin{cases} \dfrac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \\ \dfrac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha \\ \dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t} \end{cases}$$

----哈密顿正则方程

称q_α，p_α 正则变量

组成的2S维空间称相空间

$$H = H(p_\alpha, q_\alpha, t)$$

例题1：质量为m的质点做一维运动，拉氏函数为：

$$L=\frac{1}{2}me^{rt}(\dot{x}^2-\omega^2x^2)$$

试(1)由拉格朗日方程导出动力学方程;

(2)写出哈密顿函数并由哈密顿方程导出动力学方程

解： (1) $\because L=\frac{1}{2}me^{rt}(\dot{x}^2-\omega^2x^2)$

$$\therefore \frac{\partial L}{\partial \dot{x}}=me^{rt}\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=me^{rt}\ddot{x}+mre^{rt}\dot{x}$$

$$\frac{\partial L}{\partial x}=-m\omega^2e^{rt}x$$

代入保守系拉格朗日方程中得:

$$\ddot{x}+r\dot{x}+\omega^2x=0$$

解：(2) **i=1**，选广义坐标为x

$$\because p_x=\frac{\partial L}{\partial \dot{x}}=me^{rt}\dot{x} \qquad \therefore \dot{x}=\frac{p_x}{m}e^{-rt}$$

$$\therefore H=\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L=p_x\dot{x}-\frac{m}{2}e^{rt}(\dot{x}^2-\omega^2x^2)$$

$$=p_x(\frac{p_x}{m}e^{-rt})-\frac{m}{2}e^{rt}[(\frac{p_x}{m}e^{-rt})^2-\omega^2x^2]$$

$$=\frac{p_x^2}{2m}e^{-rt}+\frac{m}{2}e^{rt}\omega^2x^2$$ 写成x，p的函数

由哈密顿正则方程知：$\dot{p}_x=-\frac{\partial H}{\partial x}=-\frac{m}{2}e^{rt}\omega^2\,2x \qquad (1)$

$$\because \dot{x}=\frac{p_x}{m}e^{-rt} \qquad \therefore \ddot{x}=\frac{\dot{p}_x}{m}e^{-rt}-r\frac{p_x}{m}e^{-rt} \qquad (2)$$

(1)代入(2)中： $\ddot{x}+r\dot{x}+\omega^2x=0$

哈密顿方程的解题步骤:

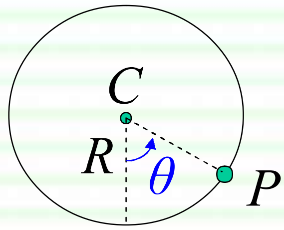
①分析受力和约束情况，选定广义坐标。写出体系动能、势能和拉格朗日函数;

②根据定义求出广义动量的表达式，并反解出广义速度的表达式;

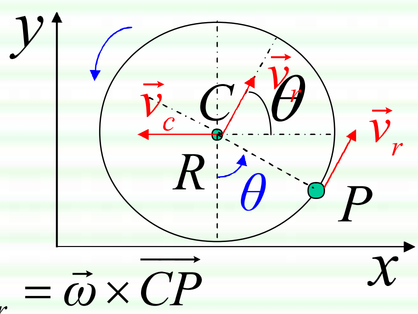
③根据定义写出正则变量表示的哈密顿函数;

④代入正则方程，得到一阶微分方程。

例题2：半径为R、质量为M的均质圆盘边缘上固定一质量为m的质点P，圆盘可在水平面上做无滑滚动。质点P与盘心C的连线与竖直线的夹角为θ，如图示。写出系统的哈密顿函数和正则方程。



解：i=1, 选广义坐标为 θ



圆盘绕质心转动速度为：$\vec{\omega} = \dot{\vec{\theta}}$

圆盘质心平动速度为：$v_c = R\dot{\theta}$

质点P的速度为：$\vec{v} = \vec{v}_c + \vec{v}_r$ $\quad \vec{v}_r = \vec{\omega} \times \overrightarrow{CP}$

$$\therefore \vec{v} = -R\dot{\theta}\vec{i} + R\dot{\theta}(\cos\theta\vec{i} + \sin\theta\vec{j}) \qquad \therefore v^2 = 2R^2\dot{\theta}^2(1-\cos\theta)$$

Or 矢量图法求：

体系动能：
$$T = (\frac{1}{2}Mv_c^2 + \frac{1}{2}I_c\omega^2) + \frac{1}{2}mv^2$$
$$= \frac{1}{2}\left[\frac{3}{2}M + 2m(1-\cos\theta)\right]R^2\dot{\theta}^2$$

体系势能：$V = -mgR\cos\theta$ $\qquad V_c = 0$

$$\therefore L=\frac{1}{2}\left[\frac{3}{2}M+2m(1-\cos\theta)\right]R^2\dot{\theta}^2+mgR\cos\theta$$

广义动量： $$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=\left[\frac{3}{2}M+2m(1-\cos\theta)\right]R^2\dot{\theta}$$

$$\therefore\dot{\theta}=\frac{p_\theta}{\left[\frac{3}{2}M+2m(1-\cos\theta)\right]R^2}$$

$$\therefore H=\sum_{\alpha=1}^{s}\frac{\partial L}{\partial\dot{q}_\alpha}\dot{q}_\alpha-L=\frac{\partial L}{\partial\dot{\theta}}\dot{\theta}-L$$

$$=\frac{1}{2}\left[\frac{3}{2}M+2m(1-\cos\theta)\right]R^2\dot{\theta}^2-mgR\cos\theta$$

$$=\frac{p_\theta^2}{2\left[\frac{3}{2}M+2m(1-\cos\theta)\right]R^2}-mgR\cos\theta$$

$$H=\frac{p_\theta^2}{2\left[\frac{3}{2}M+2m(1-\cos\theta)\right]R^2}-mgR\cos\theta$$

$$\because \dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha} \qquad \dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}$$

$$\therefore \begin{cases} \dot{\theta}=\dfrac{\partial H}{\partial p_\theta}=\dfrac{p_\theta}{\left[\frac{3}{2}M+2m(1-\cos\theta)\right]R^2} \\ \dot{p}_\theta=-\dfrac{\partial H}{\partial \theta}=\dfrac{p_\theta^2 m\sin\theta}{\left[\frac{3}{2}M+2m(1-\cos\theta)\right]^2 R^2}-mgR\sin\theta \end{cases}$$

例题3（5.20）已知一带电粒子在电磁场中的拉格朗日函数L为： $L=T-q\varphi+q\vec{A}\cdot\vec{v}=\frac{1}{2}mv^2-q\varphi+q\vec{A}\cdot\vec{v}$

试由此写出它的哈密顿函数。

解： ***i*=3**, 选广义坐标为***x, y, z***; 广义速度为： $\dot{x},\dot{y},\dot{z}$

$$\because\begin{cases}p_x=\frac{\partial L}{\partial\dot{x}}=m\dot{x}+qA_x\\ p_y=\frac{\partial L}{\partial\dot{y}}=m\dot{y}+qA_y\\ p_z=\frac{\partial L}{\partial\dot{z}}=m\dot{z}+qA_z\end{cases}\Rightarrow\begin{cases}\dot{x}=\frac{p_x-qA_x}{m}\\ \dot{y}=\frac{p_y-qA_y}{m}\\ \dot{z}=\frac{p_z-qA_z}{m}\end{cases}$$

$$\therefore H=\sum_{\alpha=1}^{3}p_\alpha\dot{q}_\alpha-L=p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-\frac{1}{2}mv^2+q\varphi-q\vec{A}\cdot\vec{v}$$

$$=p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)+q\varphi-q(A_x\dot{x}+A_y\dot{y}+A_z\dot{z})$$

$$= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q\varphi - q(A_x\dot{x} + A_y\dot{y} + A_z\dot{z})$$

$$= p_x\frac{p_x - qA_x}{m} + p_y\frac{p_y - qA_y}{m} + p_z\frac{p_z - qA_z}{m}$$

$$-\frac{1}{2}m\frac{(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2}{m^2}$$

$$+ q\varphi - q(A_x\frac{p_x - qA_x}{m} + A_y\frac{p_y - qA_y}{m} + A_z\frac{p_z - qA_z}{m})$$

$$= \frac{p^2}{2m} + q\varphi + \frac{q^2A^2 - 2q\vec{A}\cdot\vec{p}}{2m}$$

例题4（5.21）试写出自由质点在作匀速转动的坐标系中的哈密顿函数的表示式。

解：i=3, 选广义坐标为x, y, z; 广义速度为：$\dot{x}, \dot{y}, \dot{z}$

质点动能：$T=\frac{1}{2}mv^2$　　势能为：V

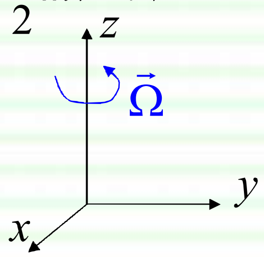
质点绝对速度为：$\vec{v}=\vec{v}_r+\vec{\Omega}\times\vec{r}$;　$\vec{v}_r=\dot{x}\vec{i}+\dot{y}\vec{j}+\dot{z}\vec{k}$

$$\therefore H=\sum_{\alpha=1}^{3}p_\alpha\dot{q}_\alpha-L=p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-\frac{1}{2}mv^2+V$$

$$=\vec{P}\cdot\vec{v}_r-\frac{1}{2}mv^2+V\qquad=\vec{P}\cdot(\vec{v}-\vec{\Omega}\times\vec{r})-\frac{1}{2}mv^2+V$$

$$=\frac{p^2}{m}-\vec{P}\cdot(\vec{\Omega}\times\vec{r})-\frac{p^2}{2m}+V$$

$$=\frac{p^2}{2m}-\vec{P}\cdot(\vec{\Omega}\times\vec{r})+V$$



作业：一、习题5.20、5.21、5.23

下次课:初积分问题