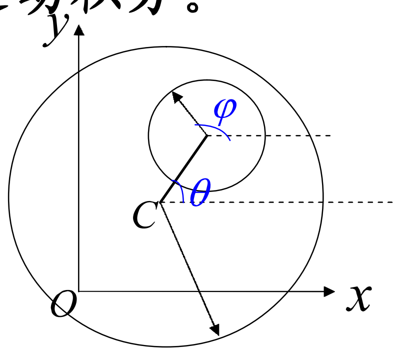
作业:桌面上一个质量为M、半径为R的均质圆盘，半径为r的均质(m)小圆盘其中心固定在大圆盘中心的距离b处。小圆盘可在大圆盘上无摩擦地转动，水平向无外力，求此系统的所有运动积分。

解: 取静止坐标系oxy，$i=4$，

广义坐标：x_c，y_c，θ和φ



$$\therefore L = T = T_M + T_m$$

$$= \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\cdot\frac{1}{2}MR^2\dot{\theta}^2$$

$$+\frac{1}{2}m\left[\left(\dot{x} - b\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + b\dot{\theta}\cos\theta\right)^2\right] + \frac{1}{2}\cdot\frac{1}{2}mr^2\dot{\varphi}^2$$

$$\therefore T=\frac{1}{2}\left[(m+M)\dot{x}^2+(m+M)\dot{y}^2\right]+\left(\frac{1}{4}MR^2+\frac{1}{2}mb^2\right)\dot{\theta}^2$$

$$-mb\dot{x}\dot{\theta}\sin\theta+mb\dot{y}\dot{\theta}\cos\theta+\frac{1}{4}mr^2\dot{\varphi}^2$$

由：$\frac{\partial L}{\partial x}=0$ 有：$\frac{\partial L}{\partial \dot{x}}=(M+m)\dot{x}-mb\dot{\theta}\sin\theta=C_1$

$\frac{\partial L}{\partial y}=0$ $\frac{\partial L}{\partial \dot{y}}=(M+m)\dot{y}+mb\dot{\theta}\sin\theta=C_2$

表示**x**和**y**向动量守恒;

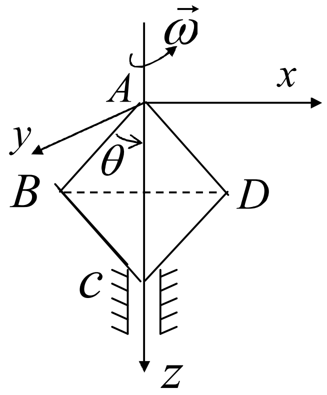
$\because \frac{\partial L}{\partial \varphi}=0$ $\therefore \frac{\partial L}{\partial \dot{\varphi}}=\frac{1}{2}mr^2\dot{\varphi}=C,\dot{\varphi}=C_3$

表示小圆盘质心角动量守恒;

$\because \frac{\partial L}{\partial t}=0,$ $T=T_2$ $\therefore T+V=C_4=T$

表示机械能守恒;

〖5.5〗在离心加速器中，质量为m_2的质点C沿着一竖直轴运动，而整个系统则以匀角速ω绕该轴转动。试写出此力学体系的拉氏函数，设连杆AB、BC、CD、DA等的质量均可不记。



解：保守力系，i=1，选广义坐标θ

各质点位置为：

$$\begin{cases} x_B = -a\sin\theta \\ z_B = a\cos\theta \end{cases} \qquad \begin{cases} \dot{x}_B = -a\cos\theta\cdot\dot{\theta} \\ \dot{z}_B = -a\sin\theta\cdot\dot{\theta} \end{cases}$$

$$\begin{cases} x_D = a\sin\theta \\ z_D = a\cos\theta \end{cases} \qquad \begin{cases} \dot{x}_D = a\cos\theta\cdot\dot{\theta} \\ \dot{z}_D = -a\sin\theta\cdot\dot{\theta} \end{cases}$$

$$\begin{cases} x_C = 0 \\ z_C = 2a\cos\theta \end{cases} \qquad \begin{cases} \dot{x}_C = 0 \\ \dot{z}_C = -2a\sin\theta\cdot\dot{\theta} \end{cases}$$

$$\because \vec{v} = \vec{v}_r + \vec{\Omega} \times \vec{r}$$
$$= (\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}) + \vec{\Omega} \times (x\vec{i} + y\vec{j} + z\vec{k})$$
$$\vec{v}_B = (\dot{x}_B\vec{i} + \dot{z}_B\vec{k}) + (-\Omega)\vec{k} \times (x_B\vec{i} + z_B\vec{k})$$
$$= \dot{x}_B\vec{i} + -\Omega x_B\vec{j} + \dot{z}_B\vec{k}$$
$$\vec{v}_D = (\dot{x}_D\vec{i} + \dot{z}_D\vec{k}) + (-\Omega)\vec{k} \times (x_D\vec{i} + z_D\vec{k})$$
$$= \dot{x}_D\vec{i} + -\Omega x_D\vec{j} + \dot{z}_D\vec{k}$$
$$\vec{v}_C = \dot{z}_C\vec{k} + (-\Omega)\vec{k} \times z_C\vec{k} = \dot{z}_C\vec{k}$$
求得： $v_B^2 = \dot{x}_B^2 + \dot{y}_B^2 + \dot{z}_B^2 = a^2\dot{\theta}^2 + \Omega^2 a^2 \sin^2\theta$
$$v_D^2 = \dot{x}_D^2 + \dot{y}_D^2 + \dot{z}_D^2 = a^2\dot{\theta}^2 + \Omega^2 a^2 \sin^2\theta$$
$$v_C^2 = \dot{z}_c^2 = 4a^2\dot{\theta}^2 \sin^2\theta$$

故体系的总动能：

$$T = T_B + T_D + T_C$$
$$= m_1(a^2\dot{\theta}^2 + \Omega^2 a^2 \sin^2\theta) + 2m_2 a^2\dot{\theta}^2 \sin^2\theta$$

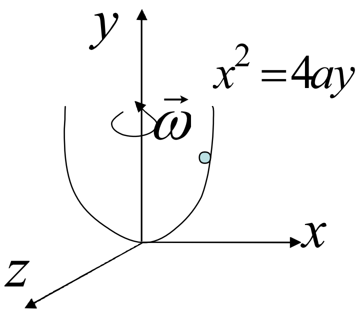
取***ox***为零势，体系的势能为：

$$V = 2V_1 + V_2 = -2m_1 ga\cos\theta - m_2 g\cdot 2a\cos\theta$$
$$= -2ga(m_1 + m_2)\cos\theta$$

故力学体系的拉氏函数：

$$L = T - V$$
$$= m_1 a^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta) + 2m_2 a^2\dot{\theta}^2 \sin^2\theta + 2ga(m_1 + m_2)\cos\theta$$

[5.7] 一个抛物线形状的光滑金属丝，取固连于它的坐标轴***oxyz***。金属丝的方程为 $x^2 = 4ay$ ，如图示，写出质点以***x***表示的运动微分方程。



解： 直接写出小环的绝对速度：

$$\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} - \omega x\vec{k}$$

$$\therefore T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \omega^2 x^2\right); \quad V = mgy$$

$$\because x^2 = 4ay \quad \therefore T = \frac{1}{2}m\left[\left(1+\frac{x^2}{4a^2}\right)\dot{x}^2 + \omega^2 x^2\right],$$

$$\therefore L = T - V = \frac{1}{2}m\left[\left(1+\frac{x^2}{4a^2}\right)\dot{x}^2 + \omega^2 x^2\right] - mg\frac{x^2}{4a}$$

$$L = T - V = \frac{1}{2}m\left[\left(1+\frac{x^2}{4a^2}\right)\dot{x}^2 + \omega^2 x^2\right] - mg\frac{x^2}{4a}$$

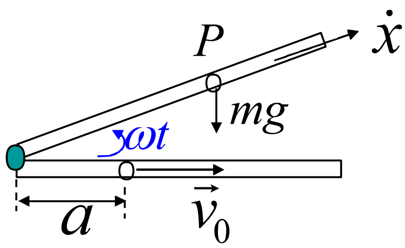
$$\because \frac{\partial L}{\partial t} = 0 \therefore T_2 - T_0 + V = Const.$$

$$T + V = \frac{1}{2}m\left[\left(1+\frac{x^2}{4a^2}\right)\dot{x}^2 + \omega^2 x^2\right] - \frac{mgx^2}{4a} \neq Const.$$

故：有广义能量积分，没有能量积分。

$$\Rightarrow m(1+\frac{1}{4a^2}x^2)\ddot{x} + \frac{m}{4a^2}x\dot{x}^2 + (\frac{g}{2a} - \omega^2)mx = 0$$

〖5.8〗一光滑细管可在竖直平面内绕通过其一端的水平轴以匀角速度ω转动，管中有一质量为m的质点，开始时，细管取水平方向，质点距转动轴的距离为a，质点相对于管的速度为v_0，试由拉格朗日方程求质点相对于管的运动规律。



解：保守力系，$i=1$，选$q=x$

质点绝对速度：$v^2 = \dot{x}^2 + \omega^2 x^2$

而动能：$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2)$

势能： $V = mgx\sin(\omega t)$ 水平方向为零势面。

而拉氏函数：

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) - mgx\sin(\omega t)$$

代入拉氏方程： $\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})-\frac{\partial L}{\partial x}=0$

得： $m\ddot{x}-m\omega^2 x=-mg\sin\omega t$

齐次方程解为：$x=c_1e^{\omega t}+c_2e^{-\omega t}$

特解为： $\frac{g}{2\omega^2}\sin\omega t$

得通解： $x=c_1e^{\omega t}+c_2e^{-\omega t}+\frac{g}{2\omega^2}\sin\omega t$

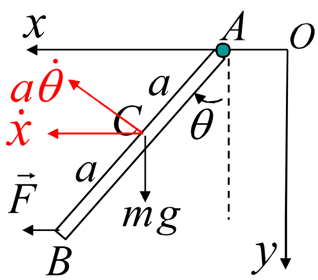
在t=0时，*x*=*a*，*v*=*v*$_0$，求出c$_1$，c$_2$：

$$c_1=\frac{1}{2}(a+\frac{v_0}{\omega})-\frac{g}{4\omega^2} \qquad c_2=\frac{1}{2}(a-\frac{v_0}{\omega})+\frac{g}{4\omega^2}$$

故方程的解为：

$$x=[\frac{1}{2}(a+\frac{v_0}{\omega})-\frac{g}{4\omega^2}]e^{\omega t}+[\frac{1}{2}(a-\frac{v_0}{\omega})+\frac{g}{4\omega^2}]e^{-\omega t}+\frac{g}{2\omega^2}\sin\omega t$$

〖5.12〗匀质棒AB，质量为m，长为2a，其a端可以在水平槽上运动，而棒本身又可以在竖直平面内绕A摆动。除有重力作用外，B端还受有一水平的力F的作用。试用拉格朗日方程求其运动微分方程。如摆动的角度很小，结果又将如何？ 5.12



解：$i=2$，选广义坐标x，θ

$$T=\frac{1}{2}mv_c^2+\frac{1}{2}I_c\omega^2$$

$$I_c=\frac{m}{12}(2a)^2=\frac{1}{3}ma^2$$

$$v_c^2=\dot{\vec{r}}^2=\dot{x}^2+a^2\dot{\theta}^2+2a\dot{\theta}\dot{x}\cos\theta$$

$$\therefore T=\frac{1}{2}m(\dot{x}^2+a^2\dot{\theta}^2+2a\dot{x}\dot{\theta}\cos\theta)+\frac{1}{2}I_c\dot{\theta}^2$$

$$\because \vec{r}_c=(x+a\sin\theta)\vec{i}+a\cos\theta\,\vec{j}$$

$$\dot{\vec{r}}_c=(\dot{x}+a\dot{\theta}\cos\theta)\vec{i}-a\dot{\theta}\sin\theta\vec{j}$$ 或者：

用虚功求广义力： $\delta W = \sum_{\alpha} Q_\alpha \delta q_\alpha = \sum_{i=1}^{2} \vec{F}_i \cdot \delta \vec{r}_i$

$$\vec{r}_B = (x + 2a\sin\theta)\vec{i} + 2a\cos\theta\vec{j}$$

$$\vec{r}_c = (x + a\sin\theta)\vec{i} + a\cos\theta\vec{j}$$

$$\therefore \delta\vec{r}_B = (\delta x + 2a\cos\theta\cdot\delta\theta)\vec{i} - 2a\sin\theta\cdot\delta\theta\vec{j}$$

$$\delta\vec{r}_C = (\delta x + a\cos\theta\cdot\delta\theta)\vec{i} - a\sin\theta\cdot\delta\theta\vec{j}$$

$$\therefore \delta W = F(\delta x + 2a\cos\theta\delta\theta) - mga\sin\theta\delta\theta$$

$$= F\delta x + (2aF\cos\theta - mga\sin\theta)\delta\theta$$

Q_x （F）　　Q_θ （$2aF\cos\theta - mga\sin\theta$）

$$\because T = \frac{1}{2}m(\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta) + \frac{1}{6}ma^2\dot{\theta}^2$$

显见$\boldsymbol{x}$微循环坐标，即： $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = Q_x$

$$\Rightarrow m(\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta) = F \qquad (1)$$

对θ带入一般形式的拉格朗日方程中:

$$\frac{4}{3}ma^2\ddot{\theta}+ma\ddot{x}\cos\theta=2Fa\cos\theta-mga\sin\theta \qquad (2)$$

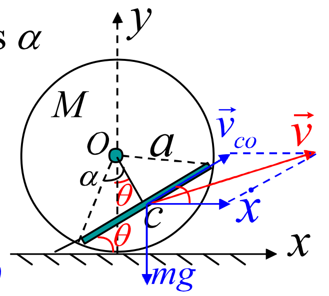
（1）（2）即为运动方程。

若摆动角很小，即：$\sin\theta\to\theta;\cos\theta\to 1$

$$\begin{cases} \ddot{x}+a\ddot{\theta}=\dfrac{F}{m} \\ \ddot{x}+\dfrac{4}{3}a\ddot{\theta}+g\theta=\dfrac{2F}{m} \end{cases}$$

〖5.15〗质量为M，半径为a的薄球壳，外表面完全粗糙，内表面完全光滑，放在粗糙水平桌上，在球壳内放一质量为m，长为2asinα的匀质棒。设此系统由静止开始运动，且在开始的瞬间，棒在通过球心的竖直平面内，两端都与球壳接触，并与水平线成β角。试用拉格朗日方程证明在以后的运动中，此棒与水平线所夹的角θ满足关系：

$$[(5M+3m)(3\cos^2\alpha+\sin^2\alpha)-9m\cos^2\alpha\cos^2\theta]a\dot{\theta}^2$$
$$=6g(5M+3m)(\cos\theta-\cos\beta)\cos\alpha$$



证明：$i=2$，选广义坐标x,θ，体系摩擦力不做功，为保守力系。

$$T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}\left(\frac{2}{3}Ma^2\right)\omega_1^2+\frac{1}{2}mv^2+\frac{1}{2}\left[\frac{m}{12}(2a\sin\alpha)^2\right]\omega_2^2$$

$$\omega_1=\frac{\dot{x}}{a},\quad \omega_2=\dot{\theta}$$

$$=\frac{5}{6}M\dot{x}^2+\frac{1}{2}m(\dot{x}^2+a^2\cos^2\alpha\dot{\theta}^2+2a\dot{x}\dot{\theta}\cos\alpha\cos\theta)+\frac{1}{6}ma^2\dot{\theta}^2\sin^2\alpha$$

$$or\qquad \vec{r}_c=(x+\overline{oc}\sin\theta)\vec{i}+\overline{oc}\cos\theta\,\vec{j}$$

或以O点为零势面

$$V=-mga\cos\alpha\cos\theta$$

或以C点为零势面

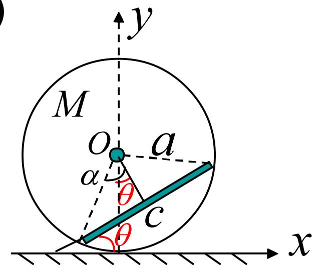
$$V=mga\cos\alpha(1-\cos\theta)$$

势能:以地面为零势面

$$V=mga(1-\cos\alpha\cos\theta)+C_1$$

$$\therefore L=\frac{1}{2}(\frac{5}{3}M+m)\dot{x}^2+\frac{1}{2}ma^2\dot{\theta}^2(\cos^2\alpha+\frac{1}{3}\sin^2\alpha)$$

$$+ma\dot{x}\dot{\theta}\cos\alpha\cos\theta-mg(a-a\cos\alpha\cos\theta)$$



显见*x*是循环坐标:

$$\therefore\frac{\partial L}{\partial\dot{x}}=(\frac{5}{3}M+m)\dot{x}+ma\dot{\theta}\cos\alpha\cos\theta=\mathrm{c}_1\qquad(1)$$

将θ代入保守系拉格朗日方程中:

$$a\ddot{\theta}(\cos^2\alpha+\frac{1}{3}\sin^2\alpha)+\ddot{x}\cos\alpha\cos\theta+g\cos\alpha\sin\theta=0\qquad(2)$$

$$(1)'\Longrightarrow(\frac{5}{3}M+m)\ddot{x}+ma\ddot{\theta}\cos\alpha\cos\theta-ma\dot{\theta}\cos\alpha\sin\theta=0\qquad(3)$$

(2)(3)联立消去*x’’*,积分可得结果

法二:

$$L=\frac{1}{2}(\frac{5}{3}M+m)\dot{x}^2+\frac{1}{2}ma^2\dot{\theta}^2(\cos^2\alpha+\frac{1}{3}\sin^2\alpha)+ma\dot{x}\dot{\theta}\cos\alpha\cos\theta$$

$$-mg(a-a\cos\alpha\cos\theta)$$

显见*x*是循环坐标:

$$\therefore\frac{\partial L}{\partial\dot{x}}=(\frac{5}{3}M+m)\dot{x}+ma\dot{\theta}\cos\alpha\cos\theta=\mathrm{c}_1 \qquad (1)$$

$$t=0,\Rightarrow\dot{\theta}=0,\theta=\beta;\Rightarrow c_1=0\ \Rightarrow\dot{x}=-\frac{ma\cos\alpha\cos\theta\cdot\dot{\theta}}{\frac{5}{3}M+m}$$

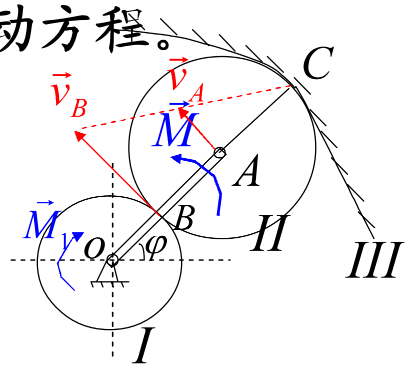
将*x*'代入L中:

$$\therefore L=[(5M+3m)(3\cos^2\alpha+\sin^2\alpha)-9m\cos^2\alpha\sin^2\theta]+ma^2\dot{\theta}^2$$

$$+mga\cos\alpha\cos\theta$$

代入保守系拉格朗日方程中并积分可得。

作业： 图示水平面内行星轮系中，长为l的曲柄OA绕O轴转动，其端点A由铰链连接一r_2的齿轮II，已知$r_2=1.5r_1$，曲柄OA质量为m，齿轮I、II的质量分别为m_1、m_2。在曲柄上作用一力偶M，齿轮I上有阻力偶M_1，求曲柄运动方程。



解： $i=1$，选曲柄转角φ为广义坐标，轮心A也是曲柄端点A。

$$\therefore v_A=(r_1+r_2)\omega=l\dot{\varphi} \qquad v_C=0$$

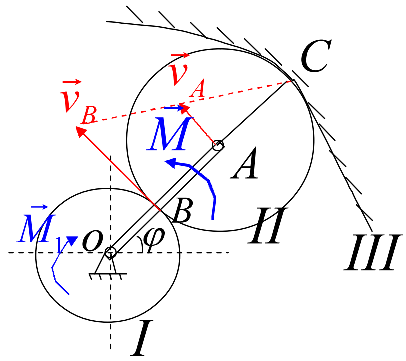
而： $$v_C=v_A-r_2\dot{\varphi}_{II}$$

则轮II的角速度：$$\dot{\varphi}_{II}=\frac{l\dot{\varphi}}{r_2}=\frac{5}{3}\dot{\varphi}$$

又轮I与轮II的啮合处B的速度：$$v_B=2v_A=2l\dot{\varphi}$$

则轮I的角速度：$$\dot{\varphi}_I=\frac{v_B}{r_1}=\frac{2v_A}{r_1}=\frac{2l\dot{\varphi}}{r_1}=5\dot{\varphi}$$

曲柄的动能:



$$T_{qb}=\frac{1}{2}J_{qbo}\dot{\varphi}^2=\frac{1}{2}\cdot\frac{1}{3}ml^2\dot{\varphi}^2=\frac{1}{6}ml^2\dot{\varphi}^2$$

齿轮I的动能:

$$T_I=\frac{1}{2}J_{Io}\dot{\varphi}_I{}^2=\frac{1}{2}\cdot\frac{1}{2}m_1r_1{}^2\dot{\varphi}_I{}^2=m_1l^2\dot{\varphi}^2$$

齿轮II的动能:

$$T_{II}=\frac{1}{2}m_2v_A^2+\frac{1}{2}J_{IIA}\dot{\varphi}_{II}{}^2$$
$$=\frac{1}{2}m_2(l\dot{\varphi})^2+\frac{1}{2}\cdot\frac{1}{2}m_2r_2^2(\frac{l\dot{\varphi}}{r_2})^2=\frac{3}{4}m_2l^2\dot{\varphi}^2$$

系统的总动能:

$$T=(\frac{1}{6}m+m_1+\frac{3}{4}m_2)l^2\dot{\varphi}^2$$

计算广义力:

给曲柄一虚位移δφ，则 $\delta\varphi_I = 5\delta\varphi$

则主动力的虚功为:

$$\delta W = M\delta\varphi - M_1\delta\varphi_I = M\delta\varphi - 5M_1\delta\varphi = \underbrace{(M - 5M_1)}_{Q_\varphi}\delta\varphi$$

代入拉格朗日方程中得:

$$(\frac{1}{3}m + 2m_1 + \frac{3}{2}m_2)l^2\ddot{\varphi} = M - 5M_1$$

$$\Rightarrow \ddot{\varphi} = \frac{M - 5M_1}{(\frac{1}{3}m + 2m_1 + \frac{3}{2}m_2)l^2}$$

设t=0时，$\varphi = \varphi_0$; $\dot{\varphi} = \dot{\varphi}_0$

$$\Rightarrow \varphi = \frac{M - 5M_1}{(\frac{1}{3}m + 2m_1 + \frac{3}{2}m_2)l^2}t^2 + \dot{\varphi}_0 t + \varphi_0$$