第十二讲 广义力的求法

Review

用虚功原理解题的步骤：

$$\delta w = 0$$

①判别约束是否理想约束；

②确定自由度并选择广义坐标；

③找出主动力及作用点；

④用广义坐标的变分表示虚位移；

⑤ 由虚功原理写出平衡方程，让每个广义坐标变分的系数为零，求解。

本次课知识点：

1、虚功原理(续)

2、广义力的求法

一、广义力

虚功的表达有两种方式:

$$\delta w = \sum_{i=1}^{3n} F_i \delta x_i \quad \text{和} \quad \delta w = \sum_{j=1}^{s} Q_j \delta q_j$$

$$\because x_i = x_i(q_1, q_2, \cdots q_s, t) \qquad \therefore \delta x_i = \sum_{j=1}^{s} \frac{\partial x_i}{\partial q_j} \delta q_j$$

$$\therefore \delta w = \sum_{i=1}^{3n} F_i \left(\sum_{j=1}^{s} \frac{\partial x_i}{\partial q_j} \delta q_j\right) = \sum_{j=1}^{s} \left(\sum_{i=1}^{3n} F_i \frac{\partial x_i}{\partial q_j}\right) \delta q_j$$

Q_j

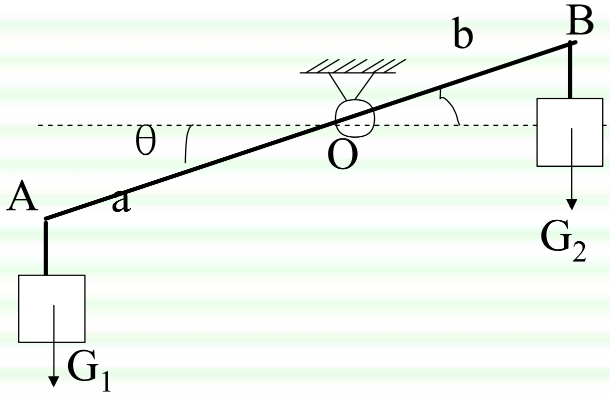
定义: $Q_j = \sum_{i=1}^{3n} F_i \frac{\partial x_i}{\partial q_j} = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ 为广义力的第 j 个分量

平衡时: $\delta w = 0$ 和 $Q_j = 0 \quad (j = 1, 2, \cdots, s)$

----虚功原理的两种表示

二、练习：

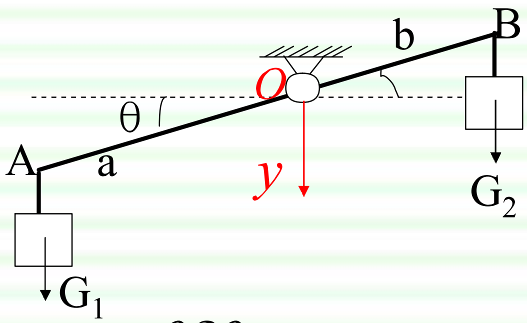
例题2： 如图示，在铅直平面内放置的杠杆AOB，支于光滑点O，臂长OA=a，OB=b，两端A、B分别挂有重量为G_1、G_2的物体，如果选择杠杆与水平方向夹角θ为广义坐标，则对应的广义力Q=?



答案： $Q=(aG_1-bG_2)\cos\theta$

解：提示

S=1，q=θ



用虚功求解:

$$\begin{cases} y_A = a\sin\theta \\ y_B = -b\sin\theta \end{cases} \Rightarrow \begin{cases} \delta y_A = a\cos\theta\delta\theta \\ \delta y_B = -b\cos\theta\delta\theta \end{cases}$$

$$\Rightarrow \delta w = G_1\delta y_A + G_2\delta y_B = (aG_1 - bG_2)\cos\theta \cdot \delta\theta$$

Q_θ

用广义力的定义求解: $Q_j = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

$$\therefore Q_\theta = \sum_{i=1}^{2} F_i \frac{\partial y_i}{\partial \theta} = G_1\frac{\partial y_A}{\partial \theta} + G_2\frac{\partial y_B}{\partial \theta} = (aG_1 - bG_2)\cos\theta$$

总结广义力的求法：

①直接用广义力的定义式：

$$Q_j=\sum_{i=1}^{3n}F_i\frac{\partial x_i}{\partial q_j}\quad or\quad Q_j=\sum_{i=1}^{n}\vec{F}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\quad (j=1,2,\cdots,s)$$

②遵照导出式的步骤，先将 δx_i 表示成 $\delta q_1,\delta q_2,\ldots,\delta q_s$ 的函数，代入 $\sum_{i=1}^{3n}F_i\delta x_i$ 中，再与 $\sum_{j=1}^{s}Q_j\delta q_j$ 比较。

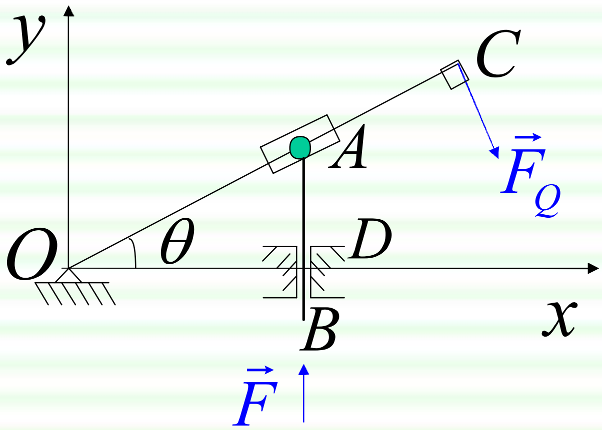
*③

练习：

①、任意质点系各广义坐标的变分都是彼此独立的。（ × ）

②、广义力一定具有力的量纲。（ × ）

例3、图示为一正切机构及其受力情况。已知***OD=l***，**OC=R**，杆件的质量及摩擦不计，**AB**杆平动，**OC**杆定轴转动。设机构于 θ**=30**^{0}的位置平衡，求力**F**与F_Q关系。

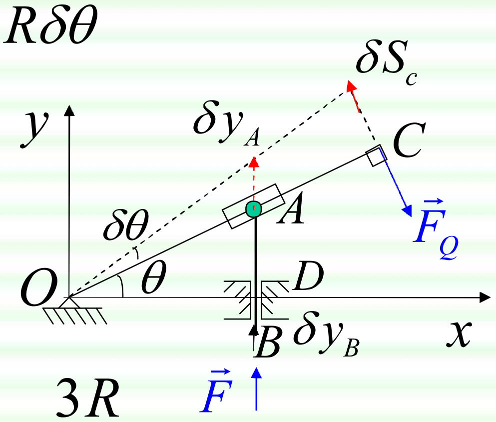


解：　$i=1$，选 θ 为广义坐标。

主动力 F 和 F_Q 给系统一虚位移 δy_B 及 δs_C，使θ角有微小增量 $\delta\theta$。

由题意知：　$\delta y_A = \delta y_B; \delta S_c = R\delta\theta$

选坐标系 Oxy，由几何关系得：



$$y_A = l tg\theta \Rightarrow \delta y_A = l\sec^2\theta \cdot \delta\theta$$

$$\because \delta W = 0 \quad \therefore F\delta y_B - F_Q \delta s_C = 0$$

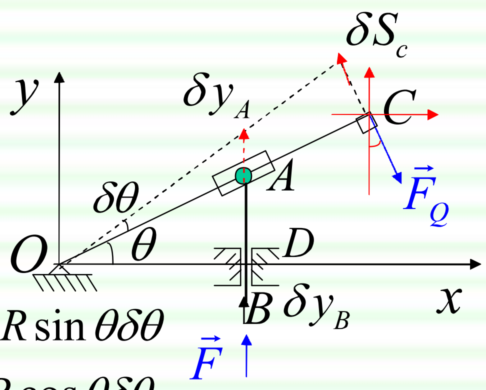
$$\frac{F}{F_Q} = \frac{\delta s_C}{\delta y_B} = \frac{R}{l}\cos^2\theta \qquad \Rightarrow \frac{F}{F_Q} = \frac{3R}{4l}$$

$\theta = 30^0$

$\delta S_C = R\delta\theta$

$\delta y_A = l\sec^2\theta\delta\theta$

或者:根据定义式:



$$Q_\theta = \sum_{i=1}^{n}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j} = \vec{F}\cdot\frac{\partial\vec{r}_A}{\partial\theta} + \vec{F}_Q\cdot\frac{\partial\vec{r}_C}{\partial\theta}$$

$$= F\frac{\partial y_A}{\partial\theta} - F_Q\frac{\partial S_C}{\partial\theta}$$

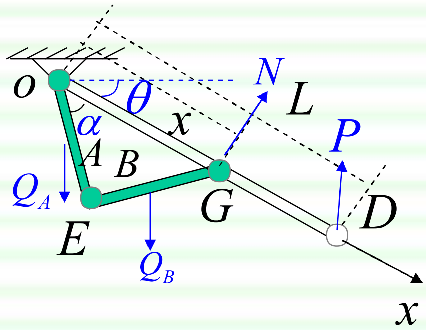
$$= Fl\sec^2\theta - F_Q R \quad \equiv 0$$

采用直角坐标: $\begin{cases} x_C = R\cos\theta \\ y_C = R\sin\theta \end{cases}$ $\begin{cases} \delta x_C = -R\sin\theta\delta\theta \\ \delta y_C = R\cos\theta\delta\theta \end{cases}$

$$Q_\theta = \vec{F}\cdot\frac{\partial\vec{r}_A}{\partial\theta} + \vec{F}_Q\cdot\frac{\partial\vec{r}_C}{\partial\theta}$$

$$= F\frac{\partial y_A}{\partial\theta} + \left(F_Q\sin\theta\right)\frac{\partial x_C}{\partial\theta} + \left(-F_Q\cos\theta\right)\frac{\partial y_C}{\partial\theta} = Fl\sec^2\theta - F_Q R$$

例4: 用光滑铰链连接两根长 l，重 Q 的均质棒A 和B，重量可忽略不计的支撑棒C 长为L，在C 棒的D端施以竖直向上的力P，所有的接触都是光滑的。**求:** **平衡**时图中坐标 x 和 θ 的值。



提示:

是否理想约束?

主动力? 约束力?

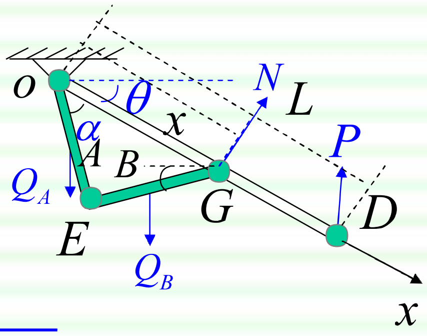
如何选择广义坐标? x、θ

虚位移? 虚功? $y_A = \dfrac{l}{2}\sin(\alpha+\theta)$ $\delta y_A = ?$

$$\begin{cases} \delta W = Q_A \delta y_A + Q_B \delta y_B + P\delta y_D \\ \delta W = Q_x \delta x + Q_\theta \delta\theta \end{cases} \Rightarrow Q_x = ?, Q_\theta = ?$$

在平衡位置处: $Q_x = 0, Q_\theta = 0 \quad \Rightarrow x = ?, \theta = ?$

解：　取$q_1=x$ 和 $q_2=\theta$　y 竖直向下



三个主动力 $\vec{Q}_A$ 、$\vec{Q}_B$ 和 $\vec{P}$

$$y_A=\frac{l}{2}\sin(\alpha+\theta)$$

而：$\cos\alpha=\frac{\frac{x}{2}}{l}=\frac{x}{2l}$，　$\sin\alpha=\sqrt{1-\frac{x^2}{4l^2}}=\frac{1}{2l}\sqrt{4l^2-x^2}$

可得：　$$y_A=\frac{1}{4}x\sin\theta+\frac{1}{4}\sqrt{4l^2-x^2}\cos\theta$$

同理：

$$y_B=x\sin\theta+\frac{l}{2}\sin(\alpha-\theta)=\frac{3}{4}x\sin\theta+\frac{1}{4}\sqrt{4l^2-x^2}\cos\theta$$

$$y_D=L\sin\theta$$

则：$$\delta y_A = \frac{1}{4}\sin\theta\delta x + \frac{1}{4}x\cos\theta\delta\theta - \frac{x}{4\sqrt{4l^2-x^2}}\cos\theta\delta x - \frac{1}{4}\sqrt{4l^2-x^2}\sin\theta\delta\theta$$

$$= \left(\frac{\sin\theta}{4} - \frac{x\cos\theta}{4\sqrt{4l^2-x^2}}\right)\delta x + \left(\frac{x\cos\theta}{4} - \frac{\sqrt{4l^2-x^2}}{4}\sin\theta\right)\delta\theta$$

$$\delta y_B = \left(\frac{3\sin\theta}{4} - \frac{x\cos\theta}{4\sqrt{4l^2-x^2}}\right)\delta x + \left(\frac{3x\cos\theta}{4} - \frac{\sqrt{4l^2-x^2}}{4}\sin\theta\right)\delta\theta$$

$\delta y_D = L\cos\theta\delta\theta$ 代入虚功原理$\delta w = 0$中：

$$\delta W = Q_A\delta y_A + Q_B\delta y_B + P\delta y_D$$

$$\therefore \delta w = \left(\theta x\cos\theta - \frac{Q}{2}\sqrt{4l^2 - x^2}\sin\theta - PL\cos\theta\right)\delta\theta$$

$$+\left(Q\sin\theta - \frac{Qx\cos\theta}{2\sqrt{4l^2 - x^2}}\right)\delta x$$

Q_θ

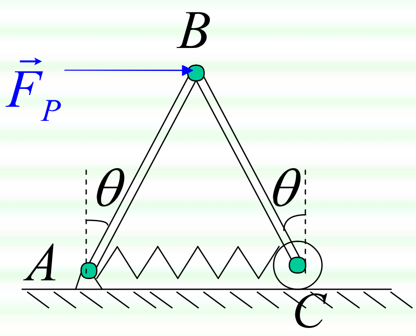
Q_x

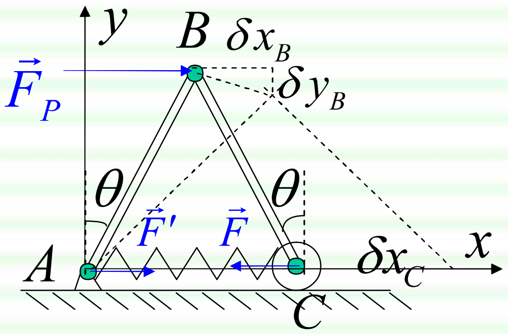
$$\because \delta w = 0 \qquad \therefore Q_x = Q_\theta = 0$$

$$\Rightarrow x = \frac{4PL}{3Q}; \qquad \theta = arctg\frac{PL}{\sqrt{(3Ql)^2 - (2PL)^2}}$$

Q_x, Q_θ? 广义力!

例5、图示所求机构(光滑接触)，$AB=BC=l$，弹簧原长为l_0，刚度为k，杆的重量忽略不计，求平衡时的角θ及弹簧张力的表达式。留为作业.





解：选坐标系Axy如图示：

把弹簧力F当作主动力。

B点的虚位移为 δx_B, δy_B,

C点的虚位移为 δx_C

虚功原理： $F_p \delta x_B - F\delta x_C = 0$

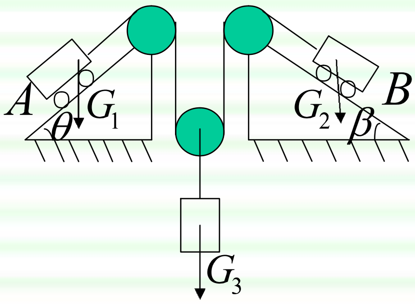
$$\begin{cases} x_B = l\sin\theta \\ x_C = 2l\sin\theta \end{cases} \Rightarrow \begin{cases} \delta x_B = l\cos\theta\cdot\delta\theta \\ \delta x_C = 2l\cos\theta\cdot\delta\theta \end{cases} \Rightarrow F = \frac{F_p}{2}$$

此时弹簧伸长： $\delta = x_C - l_0 = 2l\sin\theta - l_0$

弹簧力的大小为：

$$F = k\delta = k(2l\sin\theta - l_0) \Rightarrow \sin\theta = \frac{F_p + 2kl_0}{4kl}$$

例6: 若不计滑轮及绳的质量和摩擦，求平衡时 G_1、G_2的值。留为作业.



解: i=2?

$$G_1\delta y_A + G_2\delta y_B - G_3\delta y_3 = 0$$

$$\because 2\delta y_3 = \frac{\delta y_A}{\sin\theta} + \frac{\delta y_B}{\sin\beta}$$

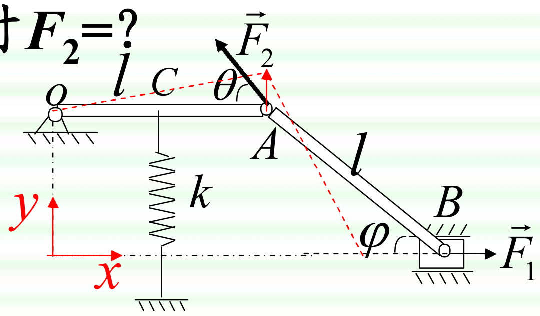
$$\therefore 2G_1\delta y_A + 2G_2\delta y_B - G_3\left(\frac{\delta y_A}{\sin\theta} + \frac{\delta y_B}{\sin\beta}\right) = 0$$

$$\therefore \left(G_1 - \frac{G_3}{2\sin\theta}\right)\delta y_A + \left(G_2 - \frac{G_3}{2\sin\beta}\right)\delta y_B = 0$$

0

$$\Rightarrow G_1 = \frac{G_3}{2\sin\theta}; G_2 = \frac{G_3}{2\sin\beta}$$

小课题、如图:已知OC=CA，F_1=200N，k=10N/cm。平衡时φ=30⁰, θ =60⁰,弹簧已有净伸长δ =2cm，OA水平。试用虚功原理求机构平衡时F_2=?



解: 弹簧力视为主动力

F=20N，系统受理想约束。

i=1，取q=φ

虚位移间关系求法①:

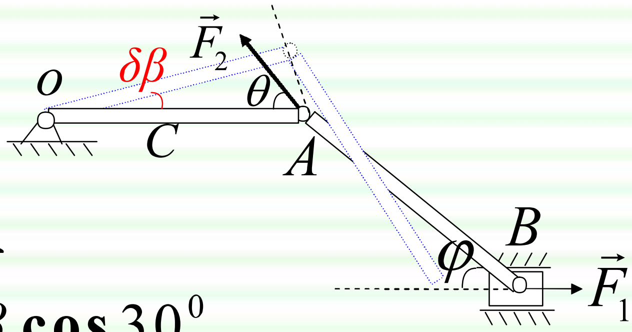
$$\because \begin{cases} y_A = l\sin\varphi \\ x_B = l + l\cos\varphi \end{cases} \quad \therefore \begin{cases} \delta y_A = l\cos\varphi\delta\varphi \\ \delta r_B = -l\sin\varphi\delta\varphi \end{cases} \quad \delta\vec{r}_A \text{ 竖直向上}$$

虚位移原理: $\delta r_c \Rightarrow \delta y_c = \dfrac{\delta y_A}{2} = \dfrac{\delta r_A}{2} = \dfrac{l}{2}\cos\varphi\delta\varphi$

$$\therefore \delta w = -F\delta r_c + F_2\cos(90^0 - \theta)\delta r_A - F_1\delta r_B = 0$$

$$F_2 \approx 144.5N$$

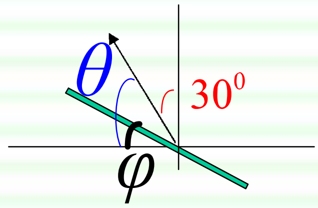
虚位移间关系求法②:



取$q=\beta$　　$\delta r_c = \frac{l}{2}\delta\beta$

$l\delta\beta$ 近似在竖直方向上

F_2方向上的位移:　$l\delta\beta\cos 30^0$



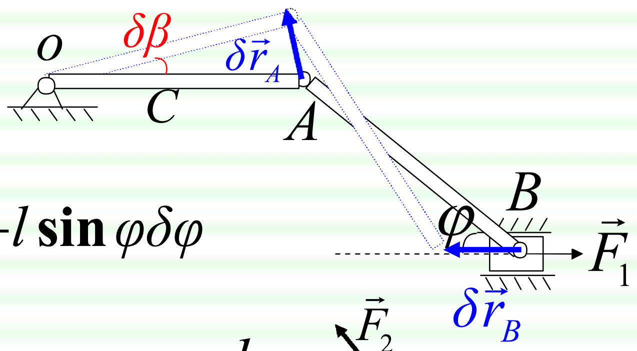
在AB方向上的位移: $l\delta\beta\cos 60^0$

$\therefore F_1$的位移为: $\dfrac{l\delta\beta\cos 60^0}{\cos\varphi}$

$$\therefore \delta w = (-F\frac{l}{2} + F_2 l\cos 30^0 - F_1 l\frac{\cos 60^0}{\cos\varphi})\ \delta\beta = 0$$

$\therefore F_2 = \frac{4}{3}(100+5\sqrt{3}) \approx 144.5$　　结果一样！！

虚位移间关系求法③:

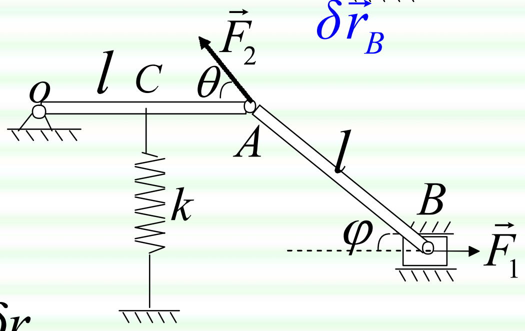


显见: $\delta r_c = \frac{1}{2}\delta r_A = \frac{1}{2} l\delta\beta$

$\because r_B = l + l\cos\varphi \therefore \delta r_B = -l\sin\varphi\delta\varphi$

$\because \delta\vec{r}_A \cdot \overrightarrow{AB} = \delta\vec{r}_B \cdot \overrightarrow{AB}$

$\therefore \delta\beta = \frac{l_1\cos\varphi}{l_2\cos\beta}\delta\varphi = \cos\varphi\delta\varphi$



$\therefore \delta r_c = \frac{l}{2}\cos\varphi\delta\varphi$

$\therefore \delta w = -F\delta r_c + F_2\sin\theta\delta r_A - F_1\delta r_B$

$= (-F\frac{l}{2}\cos\varphi + F_2\sin\theta l\cos\varphi - F_1 l\sin\varphi)\ \delta\varphi\ = 0$

$\therefore F_2 = \frac{4}{3}(100 + 5\sqrt{3}) \approx 144.5$

三、约束力的求法:

在满足理想约束的条件下，约束力在方程中不出现。

优点？缺点？

如何求解约束力?

①发挥各种理论的长处；分析力学+矢量力学

② 解除某个约束，将其变为主动力后再次利用虚功原理(保证仍为理想约束)。 如：例2

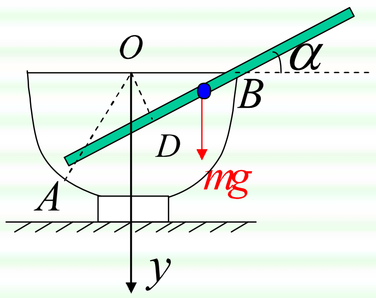
③利用达朗贝尔原理求约束力。(下次课内容)

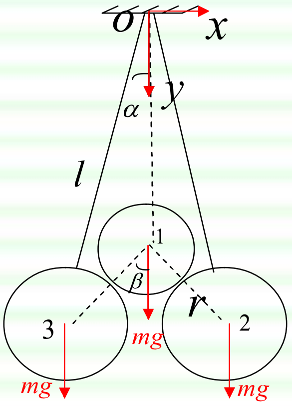
④**其它方法：如拉格朗日未定乘子法。

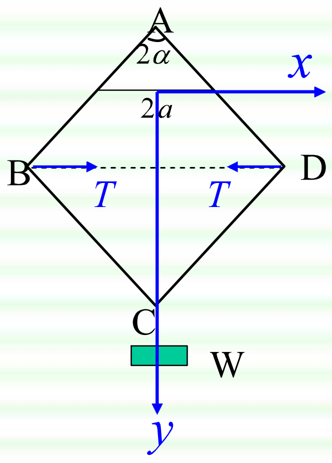
作业:

一、完成课堂习题

二:完成预留作业。







下次课内容:

虚功原理的应用(续)、约束力的求法