Chapter 11

Lecture 39 Series with function terms (I)

§ 1 Uniform convergence of series with function terms

1.1 Definition of series with function terms

The general form of a series of function terms is as follows.

$$\sum_{n=1}^{\infty} u_n(x).$$

The nth partial sum of $\sum_{n=1}^{\infty} u_n(x)$: $S_n(x)=\sum_{k=1}^{n} u_k(x)$;

Point convergence: For any fixed x_0, if $\{S_n(x_0)\}$ converges, then we call that $\sum_{n=1}^{\infty} u_n(x)$ is convergent at x_0.

Convergence in a domain: If $\{S_n(x)\}$ converges at any $x \in D$, then we call that $\sum_{n=1}^{\infty} u_n(x)$ is convergent on D.

In this case, $\sum_{n=1}^{\infty} u_n(x)$ determine a function on D, which is denoted by

$$S(x) = \sum_{n=1}^{\infty} u_n(x).$$

Example 1.1.1 $\sum_{n=0}^{\infty} x^n$ determines the function $\frac{1}{1-x}$ on $D = \{x : |x| < 1\}$.

1.2 Uniform convergence.

Example 1.2.1

Let $\sum_{n=1}^{\infty} u_n(x) = x + (x^2 - x) + \cdots + (x^{n+1} - x^n) + \cdots$, where $x \in [0,1]$.

Then $$S_n(x) = x^n,$$

and $$S(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}.$$

Remark 1.2.1 Example 1.2.1 shows that although each $u_n(x)$ is continuous on $[0,1]$, $S(x)$ may be not, and

$$\lim_{x\to 1-0} \lim_{n\to\infty} S_n(x) \neq \lim_{n\to\infty} \lim_{x\to 1-0} S_n(x).$$

Example 1.2.2 Let $S_n(x)=2n^2xe^{-n^2x^2}$.

Obviously, $\lim\limits_{n\to\infty} S_n(x)=0$ and therefore,

$$\int_0^1 S(x)dx=0.$$

But $\int_0^1 S_n(x)dx=1-e^{-n}\to 1$ (as $n\to\infty$),

That is

$$\lim_{n\to\infty}\int_0^1 S_n(x)dx \neq \int_0^1 \lim_{n\to\infty} S_n(x)dx.$$

Definition 1.2.1 For any given sequence of functions $\{S_n(x)\}$, where $x \in X$, if for any $\varepsilon > 0$, there exists some $N > 0$ which depends only on ε such that for all $n > N$,

$$|S_n(x) - S(x)| < \varepsilon,$$

then we call that the sequence $\{S_n(x)\}$ is uniformly convergent on X.

Definition 1.2.2 For a series with function terms $\sum_{n=1}^{\infty} u_n(x)$, if for any $\varepsilon > 0$, there exists some $N > 0$ which depends only on ε such that for all $n > N$,

$$\left|\sum_{k=1}^{n} u_k(x) - \sum_{k=1}^{\infty} u_k(x)\right| < \varepsilon$$

or

$$\left|\sum_{k=n+1}^{\infty} u_k(x)\right| < \varepsilon,$$

then $\{S_n(x)\}$ uniformly converges to $S(x)$ on X.

Definition 1.2.3 Define $\|S_n - S\| = \sup_{x\in X}|S_n(x) - S(x)|$.

If $\|S_n - S\| \to 0$ as $n \to \infty$, then $\{S_n(x)\}$ uniformly converges to $S(x)$ on X.

Proposition 1.2.1 Definitions 1.2.1 and 1.2.3 are equivalent.

Proof The equivalence of Definition 1.2.1 and 1.2.2 is obvious. In the following, we prove the equivalence of Definition 1.2.2 and 1.2.3.

The implication from Definition 1.2.2 to Definition 1.2.3.

For any $\varepsilon > 0$, there exists some $N > 0$ which depends only on ε such that for all $n > N$,

$$\left|\sum_{k=1}^{n} u_k(x) - S(x)\right| < \varepsilon,$$

which implies

$$\|S_n - S\| = \sup_{x \in X} |S_n(x) - S(x)| \le \varepsilon.$$

Hence

$$\|S_n - S\| \to 0 \text{ as } n \to \infty.$$

The implication from Definition 1.2.3 to Definition 1.2.2.

If $\|S_n - S\| \to 0$ as $n \to \infty$, then for any $\varepsilon > 0$, there exists some $N > 0$ which depends only on ε such that for all $n > N$,

$$\|S_n - S\| = \sup_{x\in X} |S_n(x) - S(x)| < \varepsilon.$$

This shows that

$$\left|\sum_{k=1}^{n} u_k(x) - S(x)\right| < \varepsilon.$$

The proof is finished.

Example 1.2.3 Let $S_n(x) = \dfrac{x}{1+n^2x^2}$, where $x \in (-\infty, +\infty)$. Discuss the uniform convergence of the sequence $\{S_n(x)\}$.

Solution Obviously, $S_n(x) \to S(x) = 0$. Hence

$$\|S_n - S\| = \sup_{x\in(-\infty,+\infty)} \left|\frac{x}{1+n^2x^2}\right| \le \frac{1}{2n}.$$

This shows that $\{S_n(x)\}$ uniformly converges to $S(x)=0$ on $(-\infty, +\infty)$.

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This shows that $\{S_n(x)\}$ uniformly converges to $S(x)=0$ on $(-\infty, +\infty)$.

Example 1.2.4 Let $S_n(x)=\frac{nx}{1+n^2x^2}$, where $x\in[0,1]$.

Discuss the uniform convergence of the sequence $\{S_n(x)\}$.

Solution Obviously, $S(x)=0$. This yields that

$$\|S_n - S\| = \max_{0\le x\le 1}\left|\frac{nx}{1+n^2x^2}\right| = \frac{1}{2}.$$

Hence $\|S_n - S\|$ doesn't uniformly converge to $S(x)=0$.

§ 2 Cauchy's criterion

Theorem 1.2 A sequence of functions $\{S_n(x)\}$ uniformly converges if and only if for any $\varepsilon > 0$, there exists a positive integer $N > 0$ which depends only on ε such that for all $n > N$ and $p > 0$,

$$\left|S_{n+p}(x) - S_n(x)\right| < \varepsilon.$$

Proof **Necessity** The uniform convergence of $\{S_n(x)\}$ implies that for any $\varepsilon > 0$, there exist some function $S(x)$ and $N > 0$ which depends only on ε such that for all $n > N$,

$$\left|S_n(x)-S(x)\right|<\frac{\varepsilon}{2}.$$

Since $p>0$, we see that

$$\left|S_{n+p}(x)-S(x)\right|<\frac{\varepsilon}{2}.$$

These show that

$$\left|S_{n+p}(x)-S_n(x)\right|<\varepsilon.$$

Sufficiency Since for any $\varepsilon>0$, there exists some $N>0$ which depends only on ε such that for all $n>N$ and $p>0$,

$$\left|S_{n+p}(x)-S_n(x)\right|<\varepsilon,$$

we know that for any $x \in D$, $\lim\limits_{n\to\infty} S_n(x)$ exists. Hence $S(x)$ defines a function in D.

It follows that

$$|S_n(x)-S(x)| \le |S_{n+p}(x)-S(x)| + |S_n(x)-S_{n+p}(x)| \le 2\varepsilon,$$

showing that $\{S_n(x)\}$ uniformly converges to $S(x)$.

§ 3 Properties

Theorem3.1 Suppose

(1) $S_n(x)$ is continuous on $[a,b]$;

(2) $S_n(x)$ uniformly converges to $S(x)$.

Then $S(x)$ is continuous on $[a,b]$.

Proof Since $S_n(x)$ uniformly converges to $S(x)$ on $[a,b]$, we see that for any $\varepsilon > 0$, there exists some N which depends only on ε such that for all $n > N$,

$$|S_n(x) - S(x)| < \frac{\varepsilon}{3}.$$

This implies that for any $\alpha \in [a,b]$,

$$|S_{N+1}(\alpha) - S(\alpha)| < \frac{\varepsilon}{3}.$$

Also there exists some $\eta > 0$ such that for all x: $|x - \alpha| < \eta$,

$$|S_{N+1}(x) - S_{N+1}(\alpha)| < \frac{\varepsilon}{3}.$$

We deduce that for all x: $|x-\alpha|<\eta$,

$$|S(x)-S(\alpha)| \le |S(x)-S_{N+1}(x)| + |S_{N+1}(x)-S_{N+1}(\alpha)| + |S_{N+1}(\alpha)-S(\alpha)|$$

$$<\varepsilon.$$

The proof is finished.

Theorem 3.2 Suppose

(1) $S_n(x)$ is continuous on $[a,b]$;

(2) $S_n(x)$ uniformly converges to $S(x)$.

Then

$$\lim_{n\to\infty}\int_a^b S_n(x)dx = \int_a^b \lim_{n\to\infty} S_n(x)dx.$$

Proof Since $S_n(x)$ uniformly converges to $S(x)$, we see that for any $\varepsilon > 0$, there exists some N which depends only on ε such that for all $n > N$,

$$|S_n(x) - S(x)| < \varepsilon .$$

This yields that

$$\left|\int_a^b S_n(x)dx - \int_a^b S(x)dx\right| \le \int_a^b |S_n(x) - S(x)|dx < \varepsilon(b-a),$$

which concludes the proof.

From Theorem 3.2, the following is obvious.

Corollary 3.3 Under the hypotheses of Theorem 3.2,

$$\int_a^x S_n(t)dt$$

uniformly converges to $\int_a^x S(t)dt$.

Homework Page 88: 1 (3, 4, 6); 2 (2, 4, 6); 5