Lecture 17 The list of derivation and added examples (I)

§ 1 The table of formulas of derivation

(1) $c' = 0$, where c is a constant.

(2) $(x^{\alpha})' = \alpha x^{\alpha-1} (\alpha \in R)$; $(\frac{1}{x})' = -\frac{1}{x^2}$. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.

(3) $(a^x)' = a^x \ln a$. $(e^x)' = e^x$.

(4) $(\log_a x)' = \frac{1}{x \log a}$. $(\log x)' = \frac{1}{x}$.

(5) $(\sin x)' = \cos x$. (6) $(\cos x)' = -\sin x$.

(7) $(\tan x)' = \sec^2 x$.　　(8) $(\cot x)' = -\csc^2 x$.

(9) $(\sec x)' = \sec x \tan x$.　　(10) $(\csc x)' = -\csc x \cot x$.

(11) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.　　(12) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$.

(13) $(\arctan x)' = \frac{1}{1+x^2}$.　　(14) $(\text{arccot}\, x)' = -\frac{1}{1+x^2}$.

(15) $(shx)' = chx$.　　(16) $(chx)' = shx$.

§ 2 Added examples

Examples 1.1　Find the derivatives of the following functions.

(1) $f(x)=x\arctan x - xe^x + \frac{\log x}{\sqrt{x}}$; (2) $f(x)=\frac{x\sin x+\cos x}{x\sin x-\cos x}$;

(3) $f(x)=x\sqrt{1-x}+\arcsin x$; (4) $f(x)=x^2\sin x\log x+\frac{\tan x}{x}+8$;

(5) $f(x)=\frac{5\sin x+3\tan x}{x}$.

Example 1.2 Given $f(x)=\sqrt{\frac{(1+x)\sqrt{x}}{e^x-1}}+\arcsin\frac{1-x}{\sqrt{1+x^2}}$, find $f'(1)$;

Hint We can find the derivative of $\sqrt{\frac{(1+x)\sqrt{x}}{e^x-1}}$ by taking the logarithm.

Example 1.3 Let $f(x)=e^{-|x|}$. Discuss the continuity and derivative of $f(x)$ at 0.

Hint $f(x)=\begin{cases} e^{-x}, x\ge 0 \\ e^{x}, \ x<0 \end{cases}$ and

$$\lim_{x\to 0+0}\frac{e^{-x}-1}{x}=-\lim_{x\to 0+0}\frac{e^{-x}-e^{0}}{-x}=-(e^{x})'\Big|_{x=0}=-1.$$

Example 1.4 Suppose $f(x)$ is derivable. Find the limit:

$$\lim_{t\to 0}\frac{f(a+\alpha t)-f(a+\beta t)}{t} \quad (\alpha\beta\neq 0).$$

Hint

$$\frac{f(a+\alpha t)-f(a+\beta t)}{t}=\frac{f(a+\alpha t)-f(a)}{\alpha t}\alpha-\frac{f(b+\beta t)-f(b)}{\beta t}\beta.$$

Example 1.5 Suppose $f(x)=\begin{cases} x^{\lambda}\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$ $(\lambda\in R)$. Find the domains of λ to satisfy each of the following requirements.

① $f(x)$ is continuous at 0; ② $f(x)$ is derivative at 0;
③ $f'(x)$ is continuous at 0.

Hint $\lim\limits_{x\to 0}x^{\lambda}\sin\frac{1}{x}$ (resp. $\lim\limits_{x\to 0}x^{\lambda}\cos\frac{1}{x}$) exists if and only if $\lambda>0$ and

$$f'(x)=\begin{cases} \lambda x^{\lambda-1}\sin\frac{1}{x}-x^{\lambda-2}\cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}.$$

Example 1.6 Let $f(x)=\begin{cases} x^2\sin\frac{\pi}{x}, & x<0 \\ A, & x=0 \\ ax^2+b, & x>0. \end{cases}$, where A, a and b are constants. Find A, a and b if $f(x)$ is derivable at $x=0$ and find $f'(0)$.

Hint By using the relation of derivability and one-sided derivability and the fact that a function which is derivable is continuous.

Example 1.7 Find the limit $\lim_{x\to+\infty}(k\sqrt{(x+a_1)\cdots(x+a_k)}-x)$,

where $a_1, a_2\cdots, a_k$ are constant.

Solution Let $x=\frac{1}{t}$. Then $t\to+0$ and

$$\lim_{x\to+\infty}(\sqrt[k]{(x+a_1)\cdots(x+a_k)}-x)=\lim_{t\to+0}\frac{\sqrt[k]{(1+a_1t)\cdots(1+a_kt)}-1}{t}$$

$$=\lim_{t\to+0}\frac{(1-a_1t)\cdots(1-a_kt)-1}{t}\cdot\frac{1}{p_k^{k-1}+\cdots p_k+1}$$

$$=-\frac{a_1+\cdots+a_k}{k},$$

where $p_k=\sqrt[k]{(1+a_1t)\cdots(1+a_kt)}$.

Example 1.8 Suppose $f(x)$ is defined in $(x_0-\delta,x_0+\delta)(\delta>0)$

(1) If $f(x)$ is derivable at and x_0, then

$$\lim_{h\to0}\frac{f(x_0+h)-f(x_0-h)}{2h}=f'(x_0).$$

(2) Assume the limit $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$ exists. Is $f(x)$ derivable at x_0?

Hint we know that the answer to (2) is negative by letting $f(x) = |x|$.