Lecture 36 Special class for exercises in this chapter

Example 1 (Page 7: 3(2)) Suppose $\lim\limits_{n\to\infty} x_n > 0$ exists.

Then for any sequence $\{y_n\}$, the following holds:

$$\overline{\lim_{n\to\infty}}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}} y_n .$$

Proof We have known that

$$\overline{\lim_{n\to\infty}}(x_n \cdot y_n) \le \overline{\lim_{n\to\infty}} x_n \cdot \overline{\lim_{n\to\infty}} y_n = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}} y_n .$$

It follows from

$$y_n = \frac{x_n \cdot y_n}{x_n}$$

that

$$\overline{\lim_{n\to\infty}} y_n \le \overline{\lim_{n\to\infty}} (x_n \cdot y_n) \cdot \overline{\lim_{n\to\infty}} \frac{1}{x_n} = \frac{\overline{\lim_{n\to\infty}} (x_n \cdot y_n)}{\lim_{n\to\infty} x_n},$$

showing that

$$\lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}} y_n \le \overline{\lim_{n\to\infty}} (x_n \cdot y_n).$$

Hence

$$\overline{\lim_{n\to\infty}} (x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \overline{\lim_{n\to\infty}} y_n.$$

Example 2 (Page 34: 2) Suppose the series $\sum_{m=1}^{\infty} a_m$ is obtained from $\sum_{n=1}^{\infty} u_n$ by adding brackets and every term in each bracket is nonpositive (or nonnegative).

If $\sum_{m=1}^{\infty} a_m$ is convergent, then $\sum_{n=1}^{\infty} u_n$ itself is convergent.

As an application, please discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n}.$$

Proof (1) The convergence of $\sum_{n=1}^{\infty} u_n$

Let $S_n = \sum_{i=1}^{n} u_i$,

$$\tilde{S}_m = \sum_{j=1}^{m} a_j = \left(u_1 + \cdots + u_{n_1}\right) + \cdots + \left(u_{n_{m-1}} + \cdots + u_{n_m}\right),$$

and

$$\lim_{m\to\infty} \tilde{S}_m = S.$$

For each n, there must exist some m such that $n_{m-1} \le n < n_m$.

Then

$$\left|S_n - \tilde{S}_m\right| = \left|u_{n_{m-1}} + \cdots + u_n\right| \le \left|u_{n_{m-1}} + \cdots + u_{n_m}\right|.$$

$\lim_{m\to\infty} \tilde{S}_m = S$ implies that for any $\varepsilon > 0$, there is some

$M > 0$ such that for all $m > M$,

$$\left|S_n - \tilde{S}_m\right| < \frac{\varepsilon}{2}$$

and

$$\left|u_{n_{m-1}} + \cdots + u_{n_m}\right| < \frac{\varepsilon}{2}.$$

Let $N = n_M$. Then for any $n > n_M$ implies that there exists some $m > M$ such that $n_{m-1} \le n < n_m$.

Hence for any $n > n_M$,

$$\left|S_n - S\right| \le \left|S_n - \tilde{S}_{m-1}\right| + \left|\tilde{S}_{m-1} - S < \varepsilon\right|,$$

which shows that $\sum_{n=1}^{\infty} u_n$ converges to S.

(2) The convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n}$ by using statement (1).

Let $a_m = \frac{1}{m^2} + \cdots + \frac{1}{(m+1)^2 - 1}$. By (1), it suffices to prove the convergence of $\sum_{m=1}^{\infty} (-1)^m a_m$.

Since

$$\frac{1}{m^2} + \cdots + \frac{1}{(m+1)^2 - 1} = \left[\frac{1}{m^2} + \cdots + \frac{1}{m^2 + m - 1}\right] + \left[\frac{1}{m^2 + m} + \cdots + \frac{1}{m^2 + 2m}\right],$$

we have that

$$\frac{2}{m+1} \le \frac{1}{m^2} + \cdots + \frac{1}{(m+1)^2 - 1} \le \frac{2}{k},$$

showing that $\{a_m\}$ is decreasing and $\lim_{m\to\infty} a_m = 0$.

This yields the convergence of $\sum_{m=1}^{\infty}(-1)^m a_m$, which concludes the proof.