# Lecture 35 Properties of absolutely and conditionally convergent series

## § 1 Two new series

**Definition** 1.1 For a given series

$$\sum_{n=1}^{\infty} u_n ,$$

let

$$u_n^{+} = \frac{|u_n| + u_n}{2} = \begin{cases} u_n, & u_n > 0 \\ 0, & u_n \le 0 \end{cases}$$

and

$$u_n^{-} = \frac{|u_n| - u_n}{2} = \begin{cases} 0, & u_n \ge 0 \\ -u_n, & u_n < 0 \end{cases}.$$

**Proposition 1.1** For each $n$,

$$0 \le u_n^+ \le |u_n|, \quad 0 \le u_n^- \le |u_n|$$

and

$$u_n = u_n^+ - u_n^-, \quad |u_n| = u_n^+ + u_n^-.$$

**Theorem 1.2** (1) $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if and only if both $\sum_{n=1}^{\infty} u_n^+$ and $\sum_{n=1}^{\infty} u_n^-$ are convergent;

(2) If $\sum_{n=1}^{\infty} u_n$ is conditionally convergent, then both $\sum_{n=1}^{\infty} u_n^+$ and $\sum_{n=1}^{\infty} u_n^-$ are divergent.

Proof (1) The proof follows from the Proposition 1.1.

(2) Suppose not. Then at least one of $\sum_{n=1}^{\infty} u_n^+$ and $\sum_{n=1}^{\infty} u_n^-$ converges. Without loss of generality, we may assure that $\sum_{n=1}^{\infty} u_n^+$ is convergent. Since $\sum_{n=1}^{\infty} u_n$ is conditionally convergent and

$$u_n^- = u_n^+ - u_n,$$

we see that

$$\sum u_n^-$$

is also convergent.

(1) implies that $\sum_{n=1}^{\infty} u_n$ is absolutely convergent. This is the desired contradiction.

## § 2 New series obtained by changing the positions of $u_n$

**Definition** 2.1 For a given series $\sum_{n=1}^{\infty} u_n$,

let

$$\sum_{n=1}^{\infty} u'_n$$

be a series such that for each $n$, there is some $n_k$ such that

$$u'_n = u_{n_k}$$

and $\{u_n'\} = \{u_{n_k}\}$. Then we say that $\sum_{n=1}^{\infty} u_n'$ is a series obtained by changing the positions of $u_n$.

**Theorem** 2.2 Suppose $\sum_{n=1}^{\infty} u_n$ is absolutely convergent. Then any series $\sum_{n=1}^{\infty} u_n'$ obtained by changing the positions of $u_n$ is still absolutely convergent and

$$\sum_{n=1}^{\infty} u_n' = \sum_{n=1}^{\infty} u_n.$$

Proof We divide our discussions into two cases.

Case I $\sum_{n=1}^{\infty} u_n$ is a series with nonnegative terms.

Let $S_k'$ be the $kth$ partial sum of $\sum_{n=1}^{\infty} u_n'$. Since

$$u_1' = u_{n_1},\ u_2' = u_{n_2}, \cdots,\ u_k' = u_{n_k},$$

by taking $K = \max\{n_1, n_2, \ldots, n_k\}$, we have that for all $n > K$,

$$\begin{aligned} S_k' &= u_1' + \cdots + u_k' \\ &\le u_1 + \cdots + u_n = S_n. \end{aligned}$$

This shows that for any $k$,

$$S'_k \le S = \sum_{n=1}^{\infty} u_n.$$

Hence $\sum_{n=1}^{\infty} u'_n$ is convergent and $S' \le S$.

By changing the roles of $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u'_n$ in the discussions as above, we know that

$$S' \ge S,$$

which yields $S' = S$.

Case II General case.

The hypothesis $\sum_{n=1}^{\infty} u_n$ being absolutely convergent implies that both series $\sum_{n=1}^{\infty} u_n^+$ and $\sum_{n=1}^{\infty} u_n^-$ converges. We u

$W = \sum_{n=1}^{\infty} u_n^{+}$ and $V = \sum_{n=1}^{\infty} u_n^{-}$. Then $\sum_{n=1}^{\infty} u_n = W - V$ and $\sum_{n=1}^{\infty} |u_n| = W + V$.

Case I shows that

$$\sum_{n=1}^{\infty} |u_n'| = W + V .$$

Hence $\sum_{n=1}^{\infty} u_n'$ is absolutely convergent.

Let $\sum_{n=1}^{\infty} u_n^{+'}$ and $\sum_{n=1}^{\infty} u_n^{-'}$ be the series obtained by changing the positions of $u_n^{+}$ and $u_n^{-}$, respectively.

Then

$$\sum_{n=1}^{\infty} u_n'^{+} = \sum_{n=1}^{\infty} u_n^{+} = W \quad \text{and} \quad \sum_{n=1}^{\infty} v_n'^{-} = \sum_{n=1}^{\infty} v_n^{-} = V .$$

Hence

$$\sum_{n=1}^{\infty} u'_n = \sum_{n=1}^{\infty} (u_n'^{+} - u_n'^{-})$$

$$= W - V = \sum_{n=1}^{\infty} u_n .$$

The proof is finished.

For a conditionally convergent series, the situation is quite different as the following result shows.

**Theorem** 2.3 (The Riemann's theorem) Suppose the series $\sum_{n=1}^{\infty} u_n$ is conditionally convergent. Then for any given constant $S$ which may be $\infty$, by changing the positions of $u_n$, we can obtain a series $\sum_{n=1}^{\infty} u'_n$ such that

$$\sum_{n=1}^{\infty} u'_n = S.$$

## § 3 Product of two absolutely convergent series

**Theorem** 3.1 Suppose both series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} V_n$ are absolutely convergent and $\sum_{n=1}^{\infty} u_n = U$, $\sum_{n=1}^{\infty} V_n = V$. Then $\sum_{k=1}^{\infty} u_{n_k} v_{m_k}$ is absolutely convergent and $\sum_{k=1}^{\infty} u_{n_k} v_{m_k} = UV$,

where $\sum_{k=1}^{\infty} u_{n_k}$ and $\sum_{k=1}^{\infty} v_{m_k}$ are two series obtained by changing the positions of $u_n$ and $v_n$ in $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$, respectively.

**Proof** Let $W_k = u_{n_k} v_{m_k}$ and consider the series

$$\sum_{k=1}^{\infty} |W_k|.$$

Let

$$S^*_k = \sum_{j=1}^{n} |W_j|,$$

$$p = \max\{n_1, \cdots, n_k, m_1, \cdots, m_k\},$$

$$U^*_p = \sum_{r=1}^{p} |u_r|$$

and

$$V^*_p = \sum_{r=1}^{p} |v_r|.$$

Then

$$S^*_k \le \sum_{r=1}^{p} |u_r| \cdot \sum_{r=1}^{p} |v_r|$$
$$= U^*_p | V^*_p .$$

This shows that $\sum_{k=1}^{\infty} W_k$ is absolutely convergent.

Now we come to prove $\sum_{k=1}^{\infty} W_k = UV$.

Let's consider the series

$$\sum_{n=1}^{\infty} a_n = u_1v_1 + (u_1v_2 + u_2v_2 + u_2v_1)$$
$$+ (u_1v_3 + u_2v_3 + u_3v_2 + u_3v_1) + \cdots.$$

Then $\sum_{n=1}^{\infty} a_n$ is still absolutely convergent.

Let

$$U_n = \sum_{s=1}^{n} u_s,\ V_n = \sum_{s=1}^{n} v_s \text{ and } A_n = \sum_{s=1}^{n} a_s.$$

Then

$$A_n = U_n V_n.$$

This implies that

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} U_n V_n = UV.$$

We deduce that

$$\sum_{k=1}^{\infty} W_k = UV \,.$$

**Example 3.1** For $|q| < 1$, show that $\sum_{n=1}^{\infty} nq^{n-1} = \dfrac{1}{(1-q)^2}$.

Proof Since $\sum_{n=0}^{\infty} q^n$ is absolutely convergent and

$\sum_{n=0}^{\infty} q^n = \dfrac{1}{1-q}$, the proof easily follows from Theorem 3.1.

## § 4 An added example

**Example** 4.1 Discuss the convergence of the following series.

(1) $\sum_{n=1}^{\infty} \frac{x^n}{(1+x)(1+x^2)\cdots(1+x^n)}$ $(x>0)$; (2) $\sum_{n=1}^{\infty} \int_0^{\frac{\pi}{n}} \frac{\sin x}{1+x} dx$;

Solution (1) Let $u_n = \frac{x^n}{(1+x)(1+x^2)\cdots(1+x^n)}$. Then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[ \frac{x^{n+1}}{(1+x)\cdots(1+x^n)(1+x^{n+1})} \cdot \frac{(1+x)\cdots(1+x^n)}{x^n} \right]$$

$$=\lim_{n\to\infty}\frac{x}{1+x^{n+1}}$$

$$=\begin{cases} x, & 0<x<1 \\ \frac{1}{2}, & x=1 \\ 0, & x>1 \end{cases}.$$

Hence $\sum_{n=1}^{\infty}\frac{x^n}{(1+x)(1+x^2)\cdots(1+x^n)}$ is convergent.

(2) Since

$$0\le\int_0^{\frac{\pi}{n}}\frac{\sin x}{1+x}dx\le\int_0^{\frac{\pi}{n}}\sin x dx=2\sin^2\frac{\pi}{n}$$

and

$$\lim_{n\to\infty}\frac{2\sin^2\frac{\pi}{2n}}{\frac{\pi^2}{2n^2}}=1,$$

we know from the convergence of $\sum_{n=1}^{\infty}\frac{\pi^2}{2n^2}$ that

$\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{n}}\frac{\sin x}{1+x}dx$ converges.

**Homework** Page 43: 1; 2.