Lecture 34 Series (II)

§ 3 Abel's test and Dirichlet's test

3.1 Abel's transformation

Given two sets $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$, let

$$B_1 = b_1,\ B_2 = b_1 + b_2,\ \cdots,\ B_m = b_1 + b_2 + \cdots + b_m.$$

Then

$$\sum_{i=1}^{m} a_i b_i = \sum_{i=1}^{m-1} (a_i - a_{i+1}) B_i + a_m B_m ,$$

which is the so-called Abel's transformation. Also Abel's transformation has the following form

$$\sum_{i=1}^{m} a_i b_i = -\sum_{i=1}^{m-1} (a_{i+1} - a_i) B_i + a_m B_m .$$

3.2 Abel's Lemma

Thoerem 3.2.1 Suppose that

(1) $\{a_i\}_{i=1}^m$ is monotonic; (2) $|B_i| \le M$ for each i.

Then

$$|S| = \left| \sum_{i=1}^{m} a_i \, b_i \right| \le M (|a_1| + 2|a_m|) .$$

Proof By Abel's transformation, we see that

$$|S| = \left|\sum_{i=1}^{m} a_i b_i\right| \leq M\left|\sum_{i=1}^{m-1}(a_i - a_{i+1})\right| + M|a_m| \leq M(|a_1| + 2|a_m|).$$

The proof is complete.

It easily follows from the proof of Abel's Lemma that

Corollary 3.2.2 Suppose for each i, $|B_i| \leq M$, $a_i \geq 0$ and $\{a_i\}$ is increasing. Then

$$|S| \leq Ma_1.$$

3.3 Abel's test

Theorem 3.4 Suppose that

(1) $\sum_{n=1}^{\infty} b_n$ converges;

(2) $\{a_n\}$ is bounded, i.e., there is some $M > 0$ such that for each n, $|a_n| \le M$.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof Abel's Lemma yields that

$$\sum_{k=n+1}^{n+m} a_k b_k = \sum_{i=1}^{m} a_{n+i} b_{n+i} .$$

Since $\sum_{n=1}^{\infty} b_n$ converges, we know that for any $\varepsilon > 0$, there is some $N > 0$ such that for any $n > N$ and any $p > 0$,

$$\left| b_{n+1} + \cdots + b_{n+p} \right| < \varepsilon.$$

Hence for any $n > N$ and any $m > 0$, we have that

$$\left| \sum_{k=n+1}^{n+m} a_k b_k \right| < \varepsilon(|a_{n+1}| + 2|a_{n+m}|) \leq 3M\varepsilon .$$

This shows that $\sum_{n=1}^{\infty} a_n b_n$ converges.

3.4 Dirichlet's test

Theorem 3.4.1 Suppose that

(1) the partial sum $\{B_n\}$ of $\sum_{n=1}^{\infty} b_n$ is bounded, i.e., there exists some $M>0$ such that for each n, $|B_n| \le M$;

(2) $\{a_n\}$ is monotonic and $\lim_{n\to\infty} a_n = 0$.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof Let's consider the sum $\sum_{k=1+n}^{n+m} a_k b_k$. Since $\lim_{n\to\infty} a_n = 0$, we see that for any $\varepsilon > 0$, there must exist some N such that for any $n > N$, $|a_n| < \varepsilon$. The hypothesis (1) implies that for any $p > 0$,

$$|b_{n+1} + \cdots + b_{n+p}| = |B_{n+p} - B_n| \le 2M .$$

Hence Abel's Lemma shows the following:

$$\left|\sum_{k=1+n}^{n+m} a_k b_k\right| \le 2M\,(|a_{n+1}| + 2|a_{n+m}|) < 6M\varepsilon.$$

This finishes the proof of this theorem.

3.5 Some remarks

Remark 3.5.1 Abel's test can be derived from Dirichlet's test.

The proof is easily followed from the equality:

$$\sum_{n=1}^{\infty}(a_n - a)b_n + a\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n b_n.$$

Remark 3.5.2 Leibniz's test is a special case of Dirichlet's test. This is obvious.

3.6 Examples

Suppose $\sum_{n=1}^{\infty} u_n$ converges. Show that all series $\sum_{n=1}^{\infty} \frac{u_n}{n}$, $\sum_{n=1}^{\infty} \frac{u_n}{\sqrt{n}}$, $\sum_{n=1}^{\infty} \frac{nu_n}{n+1}$ are convergent.

Solution Since $\sum_{n=1}^{\infty} \frac{nu_n}{n+1} = \sum_{n=1}^{\infty} u_n - \sum_{n=1}^{\infty} \frac{1}{n+1} u_n$,

by Dirichlet's test, all $\sum_{n=1}^{\infty} \frac{u_n}{n}$, $\sum_{n=1}^{\infty} \frac{u_n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{nu_n}{n+1}$ are convergent.

Example 3.6.2 Suppose $\{a_n\}$ is monotonic and $\lim_{n\to\infty} a_n = 0$. Then both series $\sum_{n=1}^{\infty} a_n \sin nx$ and $\sum_{n=1}^{\infty} a_n \cos nx \quad (x \neq 2k\pi)$ are convergent.

Proof Since $\{a_n\}$ is monotonic and $\lim_{n\to\infty} a_n = 0$, it follows from the following inequalities:

$$\left|\sum_{k=1}^{n} \sin kx\right| = \frac{\left|\cos\frac{x}{2} - \cos\frac{2n+1}{2}x\right|}{\left|2\sin\frac{x}{2}\right|} \leq \frac{1}{\left|\sin\frac{x}{2}\right|} \quad (x \neq 2k\pi),$$

$$\left|\sum_{k=1}^{n}\cos kx\right| \le \frac{1}{\left|\sin\frac{x}{2}\right|} \quad (x \ne 2k\pi)$$

and Dirichlet's test that both

$$\sum_{n=1}^{\infty} a_n \sin nx \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \cos nx$$

are convergent.

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