Lecture 33 Series (I)

§ 1 Absolutely convergent series

1.1 Definition

$\sum u_n$ is called absolutely convergent if $\sum |u_n|$ is convergent.

If $\sum u_n$ is convergent but $\sum |u_n|$ is divergent, then we call $\sum u_n$ conditionally convergent.

For example, we have known that $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is convergent, but $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Hence $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}$ is conditionally convergent.

1.2 The relation between the convergence and the absolute convergence of series

Theorem 1.2.1 If $\sum u_n$ is absolutely convergent, then $\sum u_n$ itself convergent. The converse does not hold.

Proof (1) Since $\sum u_n$ is absolutely convergent, we see that for any $\varepsilon > 0$, there is some $N > 0$ such that for all $n > N$ and $p > 0$,

$$\left| u_{n+1} + \cdots + u_{n+p} \right| < \varepsilon .$$

It follows from

$$\left| u_{n+1} + \cdots + u_{n+p} \right| < \left| u_{n+1} \right| + \cdots + \left| u_{n+p} \right|$$

and Cauchy's convergence principle that $\sum u_n$ is convergent.

(2) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ is convergent, but $\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Example 1.2.1 Discuss the convergence of $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}x^n$.

Solution It follows from D'Alembert's test that $\sum_{n=1}^{\infty}\frac{1}{n}x^n$ is convergent if $|x|<1$. If $|x|>1$, then $\lim_{n\to\infty}(-1)^n\frac{1}{n}x^n\neq 0$, which implies that $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}x^n$ is divergent. If $x=1$, then $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}x^n$ converges;

if $x=-1$, then $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}x^n$ diverges.

Hence $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}x^{n}$ is absolutely convergent if $|x|<1$,

conditionally convergent if $x=1$ and divergent if $|x|>1$

or $x=-1$.

§ 2 Alternating series

2.1 Definition

$\sum a_n$ is called alternating if for each n, $a_n=(-1)^n u_n$, where

$u_n>0$.

2.2 Leibuniz's test

Theorem 2.2.1 Suppose $\sum_{n=1}^{\infty}(-1)^{n+1}u^{n}$ satisfies the following:

(1) $\{u_n\}$ is decreasing;

(2) $\lim_{n\to\infty} u_n = 0$.

Then

(1) $\sum_{n=1}^{\infty}(-1)^{n+1}u^{n}$ converges;

(2) $Sgn(r_n) = Sgn((-1)^n)$ or $r_n = 0$;

(3) $|r_n| \le u_{n+1}$.

Proof (1) Let S_n be the nth partial sum of $\sum_{n=1}^{\infty}(-1)^{n+1} u^n$.

That means

$$S_n = \sum_{n=1}^{\infty}(-1)^{k+1} u_k.$$

Now we consider two subsequences:

$\{S_{2m}\}$ and $\{S_{2m+1}\}$ of $\{S_n\}$.

For $\{S_{2m}\}$, we have that

$$S_{2m+2} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m+1} - u_{2m+2})$$

$$= S_{2m} + u_{2m+1} - u_{2m+2} \geq S_{2m}.$$

This shows that $\{S_{2m}\}$ is increasing.

On the other hand,

$$S_{2m} = u_1 - (u_2 - u_3) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m} \le u_1.$$

It follows that $\lim_{m\to\infty} S_{2m}$ exists.

Since $S_{2m+1} = S_{2m} + u_{2m+1}$, we see that $\lim_{m\to\infty} S_{2m+1}$ exists and

$\lim_{m\to\infty} S_{2m} = \lim_{m\to\infty} S_{2m+1}$. Hence $\{S_n\}$ converges.

(2) It is obvious that $r_n = \sum_{k=n+1}^{\infty} (-1)^{k+1} u_k = (-1)^n \sum_{k=n+1}^{\infty} (-1)^{k-n-1} u_k$

and $0 \le \sum_{k=n+1}^{\infty} (-1)^{k-n-1} u_k \le u_{n+1}$, which implies when $r_n \neq 0$,

$$\operatorname{sgn}\{r_n\} = \operatorname{sgn}\{(-1)^n u_{n+1}\} = \operatorname{sgn}\{(-1)^n\}$$

and $$|r_n| \le u_{n+1}.$$

These conclude the proof.

Examples 2.2.1 Discuss the convergence of the following series.

(1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ $(s>0)$; (2) $\sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n^s}$ $(s>0,\ \alpha>0)$.

(3) $\sum_{n=1}^{\infty} \sin\left(\pi\sqrt{n^2+a^2}\right)$ $(a \neq 0)$.

Solution (1) Obviously, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ is absolutely convergent when $s>1$, conditionally convergent when $0<s\le 1$.

(2) Let $a_n=\frac{(-\alpha)^n}{n^s}$. Then

$$\frac{a_{n+1}}{a_n}=\frac{(-\alpha)^{n+1}}{(n+1)^s}\cdot\frac{n^s}{(-\alpha)^n}=-\alpha\left(\frac{n}{n+1}\right)^s.$$

This shows that $\sum_{n=1}^{\infty}\frac{(-\alpha)^n}{n^s}$ is absolutely convergent when

$\alpha < 1$. If $\alpha = 1$, $\sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n^s}$ is absolutely convergent when $s > 1$, conditionally convergent when $0 < s \le 1$.

If $\alpha > 1$, we see from

$$\lim_{n \to \infty} \frac{\alpha^n}{n^s} = +\infty$$

that $\sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n^s}$ is divergent.

(3) Since

$$\sin\left(\pi\sqrt{n^2 + a^2}\right) = (-1)^n \sin\left(\pi\sqrt{n^2 + a^2} - n\pi\right)$$

$$=(-1)^n \sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}$$

and

$$\lim_{n\to\infty}\frac{a^2\pi}{\sqrt{n^2+a^2}+n}=0,$$

we know that for sufficiently large n,

$$\sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}>0$$

showing that $\sum_{n=K}^{\infty}\sin\left(\pi\sqrt{n^2+a^2}\right)$ is alternating.

Obviously, $\sin\frac{a^2\pi}{\sqrt{n^2+a^2}+n}$ monotonically goes to 0,

which tells us that $\sum_{n=1}^{\infty}\sin\left(\pi\sqrt{n^2+a^2}\right)$ is convergent.

Obviously, it is conditionally convergent.

§ 3 Added examples

Examples 3.1 Suppose $\lim_{n\to\infty}\left(n^{2n\sin\frac{1}{n}}\cdot a_n\right)=1$ and $a_n\geq 0$.

Discuss the convergence of the series $\sum_{n=1}^{\infty}a_n$.

Solution Let $\varepsilon = \frac{1}{2}$. Then there is some $N > 0$ such that for all $n > N$,

$$\frac{1}{2} < n^{2n\sin\frac{1}{n}} \cdot a_n < \frac{3}{2},$$

showing that

$$\frac{1}{2n^{2n\sin\frac{1}{n}}} < a_n < \frac{3}{2n^{2n\sin\frac{1}{n}}}.$$

Since

$$\lim_{n\to\infty} \frac{\frac{1}{n^{2n\sin\frac{1}{n}}}}{\frac{1}{n^2}} = 1,$$

we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2n\sin\frac{1}{n}}}$$

is convergent. Hence $\sum_{n=1}^{\infty} a_n$ is convergent.

Examples 3.2 Suppose $\sum_{n=1}^{\infty} u_n$ is a divergent series with nonnegative terms. Let $S_n = u_1 + u_2 + \cdots + u_n$.

Then $\sum_{n=1}^{\infty} \frac{u_n}{S_n}$ is still divergent.

Proof Since

$$\sum_{k=n+1}^{n+p} \frac{u_k}{S_k} \geq \frac{\sum_{k=n+1}^{n+p} u_k}{S_{n+p}} = \frac{S_{n+p} - S_n}{S_{n+p}} = 1 - \frac{S_n}{S_{n+p}}$$

and $\lim_{n \to \infty} S_n = +\infty$, we see that for sufficiently large p,

$$0 < \frac{S_n}{S_{n+p}} < \frac{1}{2}.$$

Hence

$$\sum_{k=n+1}^{n+p} \frac{u_k}{S_k} > \frac{1}{2}.$$

This implies that $\sum_{n=1}^{\infty} \frac{u_n}{S_n}$ is divergent.

Homework Page 34: 3 (1, 3); 4