Lecture 32 Series of nonnegative terms (II)

§2 Criteria for convergence (II)

Theorem 2.9 (the Cauchy's Test by integrals) Suppose

(1) $\sum_{n=1}^{\infty} u_n$ is a series of nonnegative terms;

(2) $\{u_n\}$ is decreasing;

(3) there is a decreasing function $f(x)$ such that $f(n) = u_n$.

Then $\sum_{n=1}^{\infty} u_n$ is convergent (resp. divergent) if and only if the sequence $\{A_n\}$ is convergent (resp. divergent),

where $A_n = \int_1^n f(x)\,dx$.

Proof It follows from

$$u_{k-1} = \int_{k-1}^{k} u_{k-1}dx \geq \int_{k-1}^{k} f(x)\,dx \geq \int_{k-1}^{k} u_k dx = u_k$$

that

$$\sum_{k=2}^{\infty} u_{k-1} \geq \sum_{k=2}^{n} \int_{k-1}^{k} f(x)\,dx = \int_1^n f(x)\,dx \geq \sum_{k=2}^{\infty} u_k.$$

The proof is finished.

Example 2.2 ($p-$series) Discuss the convergence of the following series:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\quad (p>0).$$

Solution Let $f(x)=\dfrac{1}{x^p}$. If $p\neq 1$, then

$$\lim_{n\to+\infty}\int_1^n \frac{1}{x^p}dx=\frac{1}{1-p}\lim_{n\to+\infty}\left(n^{1-p}-1\right)$$

$$=\begin{cases}\dfrac{1}{p-1}, & p>1\\ \infty, & p<1\end{cases}.$$

Hence $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent when $p>1$ and divergent when $p\le 1$.

Theorem 2.10 Suppose

(1) $\sum_{n=1}^{\infty} u_n$ is a series of nonnegative terms;

(2) $\lim_{n\to\infty} n\left(1-\frac{u_{n+1}}{u_n}\right)=r$.

(I) If $r>1$, then $\sum_{n=1}^{\infty} u_n$ is convergent;

(II) If $r<1$, then $\sum_{n=1}^{\infty} u_n$ is divergent.

Example 2.3 Find the condition under which the following series is convergent:

$$\sum_{n=0}^{\infty} \frac{n!n^{-p}}{q(q+1)\cdots(q+n)} \quad (p>0, q>0).$$

Solution It follows from

$$\frac{u_{n+1}}{u_n} = \frac{q(q+1)\cdots(q+n)(n+1)!(n+1)^{-p}}{q(q+1)\cdots(q+n)(q+n+1)n!n^{-p}} = \frac{1}{(1+\frac{q}{n+1})\left(1+\frac{1}{n}\right)^p}$$

that

$$\lim_{n\to\infty} n\left(1-\frac{u_{n+1}}{u_n}\right)=\lim_{n\to\infty}\frac{\left(1-\dfrac{1}{\left(1+\dfrac{1}{n}\cdot\dfrac{q}{1+\dfrac{1}{n}}\right)\left(1+\dfrac{1}{n}\right)^{p}}\right)}{\dfrac{1}{n}}$$

$$= \lim_{x\to 0} \frac{\left(1+\frac{qx}{1+x}\right)(1+x)^p - 1}{x\left(1+\frac{qx}{1+x}\right)(1+x)^p}$$

$$= p+q.$$

Hence the condition $p+q>1$ assures the convergence of

$$\sum_{n=0}^{\infty} \frac{n!n^{-p}}{q(q+1)\cdots(q+n)}.$$

§ 3 Added examples

Example 3.1 Discuss the convergence of the following series.

(1) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ $(p>0)$; (2) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$.

Solution (1) Since

$$\lim_{x \to +\infty} \frac{x}{(\log x)^p} = +\infty,$$

we see that for sufficiently large n,

$$\frac{1}{(\log n)^p} > \frac{1}{n},$$

we easily know from the divergence of $\sum_{n=2}^{\infty} \frac{1}{n}$ that

$\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ is divergent either.

(2) Since

$$\lim_{n \to \infty} \frac{\left(\frac{n}{2n+1}\right)^n}{\left(\frac{1}{2}\right)^n} = \frac{1}{\sqrt{e}},$$

we know from the fact $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ being convergent that $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$ is convergent.

Example 3.2 Discuss the convergence of the following series.

(1) $\sum_{n=2}^{\infty}\frac{1}{n\log n}$; (2) $\sum_{n=2}^{\infty}\frac{1}{n\log^2 n}$;

(3) $\sum_{n=9}^{\infty}\frac{1}{n\log n(\log\log n)^{1+\alpha}}$ $(\alpha>0)$.

Solution (1) Since

$$\lim_{n\to\infty}\int_2^n \frac{1}{x\log x}dx = \lim_{n\to\infty}\int_2^n \frac{1}{\log x}d\log x = \lim_{n\to\infty}\log\log x\Big|_2^n = +\infty,$$

$\sum_{n=2}^{\infty}\frac{1}{n\log n}$ is divergent.

(2) Since

$$\lim_{n\to\infty}\int_2^n \frac{1}{x\log^2 x}dx = \lim_{n\to\infty}\int_2^n \frac{1}{\log^2 x}d\log x = -\lim_{n\to\infty}\frac{1}{\log x}\Big|_2^n = \frac{1}{\log 2},$$

$\sum_{n=2}^{\infty}\frac{1}{n\log^2 n}$ is convergent.

(3) Let $f(x)=\dfrac{1}{x\log x(\log\log x)^{1+\alpha}}$.

It easily follows from Theorem 2.9 that

$$\sum_{n=9}^{\infty}\frac{1}{n\log n(\log\log n)^{1+\alpha}}$$

is convergent.

Example 3.3 Show that $\sum_{n=2}^{\infty}\frac{1}{\log n}\sin\frac{1}{n}$ is divergent.

Solution Since

$$\lim_{n\to\infty}\frac{\frac{1}{\log n}\sin\frac{1}{n}}{\frac{1}{n\log n}}=1,$$

we know from the fact $\sum_{n=2}^{\infty}\frac{1}{n\log n}$ is divergent that $\sum_{n=2}^{\infty}\frac{1}{\log n}\sin\frac{1}{n}$ is divergent either.

Example 3.4 Discuss the convergence of the following series.

(1) $\sum_{n=2}^{\infty}\frac{1}{\log n!}$; (2) $\sum_{n=9}^{\infty}\frac{1}{(\log\log n)^{\log n}}$;

(3) $\sum_{n=2}^{\infty}\frac{1}{a^{\log n}}$ $(a>0)$.

Solution (1) Since

$$\frac{1}{\log n!}=\frac{1}{\sum_{k=2}^{n}\log k}>\frac{1}{n\log n},$$

the divergence of the series $\sum_{n=2}^{\infty}\frac{1}{n\log n}$ shows that

$\sum_{n=2}^{\infty}\frac{1}{\log n!}$ is divergent.

(2) Since for sufficiently large n,

$$\frac{1}{(\log\log n)^{\log n}} = \frac{1}{e^{\log n(\log\log\log n)}} = \frac{1}{n^{\log\log\log n}} < \frac{1}{n^2},$$

we know that $\sum_{n=9}^{\infty} \frac{1}{(\log\log n)^{\log n}}$ is convergent.

(3) Since

$$a^{\log n} = e^{(\log n)\log a} = n^{\log a},$$

we see when $a > e$, $\sum_{n=2}^{\infty} \frac{1}{a^{\log n}}$ is convergent, when

$0 < a \leq e$, $\sum_{n=2}^{\infty} \frac{1}{a^{\log n}}$ is divergent.

Examples 3.5 Suppose $\sum_{n=1}^{\infty} u_n$ is a convergent series with nonnegative terms. Then $\sum_{n=2}^{\infty} \frac{\sqrt{u_n}}{\sqrt{n}\log n}$ is convergent.

Proof Since

$$\frac{\sqrt{u_n}}{\sqrt{n}\log n} \le \frac{\left(\sqrt{u_n}\right)^2 + \frac{1}{\left(\sqrt{n}\log n\right)^2}}{2},$$

and both $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=2}^{\infty} \frac{1}{n\log^2 n}$ are convergent, we get that $\sum_{n=2}^{\infty} \frac{\sqrt{u_n}}{\sqrt{n} \log n}$ is convergent.

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