Lecture 31 Series of nonnegative terms (I)

§ 1 Concept of series of nonnegative terms

Definition 1.1 For a series $\sum_{n=1}^{\infty} u_n$, if for each $n \geq 1$, $u_n \geq 0$, then $\sum_{n=1}^{\infty} u_n$ is called a series with nonnegative terms.

Proposition 1.2 Let $S_n = \sum_{k=1}^{n} u_k$. Then $\sum_{n=1}^{\infty} u_n$ is convergent if and only if $\{S_n\}$ is bounded. If $\{S_n\}$ is bounded, then $\sum_{n=1}^{\infty} u_n = +\infty$.

§2 Criteria for convergence

Theorem 2.1 (Comparison theorem) Suppose both $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two series of nonnegative terms, and suppose there is a positive constant c such that

$$u_n \leq c v_n \quad (n = 1, 2, \cdots).$$

(1) If $\sum_{n=1}^{\infty} v_n$ is convergent, then $\sum_{n=1}^{\infty} u_n$ must be convergent;

(2) If $\sum_{n=1}^{\infty} u_n$ is divergent, then $\sum_{n=1}^{\infty} v_n$ must be divergent.

The proof easily follows from Proposition 1.2.

Theorem 2.2 (The limit form of comparison theorem) For two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ of nonnegative terms, if

$$\lim_{n\to\infty}\frac{u_n}{v_n}=l,$$

where $0<l<\infty$. Then $\sum_{n=1}^{\infty} u_n$ is convergent (resp. divergent if and only if $\sum_{n=1}^{\infty} v_n$ is convergent (resp. divergent).

Proof It follows from

$$\lim_{n\to\infty}\frac{u_n}{v_n}=l$$

that for $\varepsilon_0 = \frac{l}{2}$, there is some N such that for all $n > N$

$$-\varepsilon_0 < \frac{u_n}{v_n} - l < \varepsilon_0 .$$

This yields that

$$\frac{1}{2} l v_n < u_n < \frac{3}{2} l v_n .$$

The conclusion follows from Theorem 2.1.

Corollary 2.3 For two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ of nonnegative terms, suppose

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 0 .$$

If $\sum_{n=1}^{\infty} v_n$ is convergent, then $\sum_{n=1}^{\infty} u_n$ is convergent; If $\sum_{n=1}^{\infty} u_n$ is divergent, then $\sum_{n=1}^{\infty} v_n$ is divergent.

Corollary 2.4 For two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ of nonnegative terms, suppose

$$\lim_{n\to\infty} \frac{u_n}{v_n} = +\infty .$$

If $\sum_{n=1}^{\infty} u_n$ is convergent, then $\sum_{n=1}^{\infty} v_n$ is convergent; If $\sum_{n=1}^{\infty} v_n$ is divergent, then $\sum_{n=1}^{\infty} u_n$ is divergent.

Theorem 2.5 (The Cauchy's test) Suppose $\sum_{n=1}^{\infty} u_n$ is a series of nonnegative terms. If there is some N such that for all $n > N$,

$$\sqrt[n]{u_n} \leq q < 1,$$

then $\sum_{n=1}^{\infty} u_n$ is convergent; if for any N, there is some $n > N$ such that

$$\sqrt[n]{u_n} \geq 1,$$

then $\sum_{n=1}^{\infty} u_n$ is divergent.

Proof   It follows from

$$\sqrt[n]{u_n} \le q$$

that for all $n > N$,

$$0 \le u_n \le q^n .$$

Theorem 2.1 shows that $\sum_{n=1}^{\infty} u_n$ is convergent.

If for any N, there is some $n > N$ such that

$$\sqrt[n]{u_n} \ge 1,$$

then $\lim_{n \to \infty} u_u \ne 0$. This shows that $\sum_{n=1}^{\infty} u_n$ is divergent.

Theorem 2.6 (The limit form of Cauchy's test) For a series $\sum_{n=1}^{\infty} u_n$ of nonnegative terms,

(1) if $\overline{r} = \overline{\lim_{n\to\infty}} \sqrt[n]{u_n} < 1$, then $\sum_{n=1}^{\infty} u_n$ is convergent;

(2) if $\overline{r} = \underline{\lim_{n\to\infty}} \sqrt[n]{u_n} > 1$, then $\sum_{n=1}^{\infty} u_n$ is divergent;

(3) there is some convergent series with nonnegative terms such that $\overline{r} = 1$ and divergent series with nonnegative terms for which $\underline{r} = 1$.

Proof (1) Since $\overline{r} < 1$, there is some $\varepsilon_0 > 0$ such that

$$0 < \overline{r} + \varepsilon_0 < 1.$$

Then there is some $N>0$ such that for all $n>N$,

$$0 \le \sqrt[n]{u_n} < \overline{r} + \varepsilon_0 < 1.$$

Theorem 2.5 shows that $\sum_{n=1}^{\infty} u_n$ converges.

Since $\underline{r} > 1$, there is some $\varepsilon_0 > 0$ such that

$$\underline{r} - \varepsilon_0 > 1.$$

Then there is some $N>0$ such that for all $n>N$,

$$\sqrt[n]{u_n} \ge \underline{r} - \varepsilon_0 > 1.$$

Hence $\lim_{n\to\infty} u_u \ne 0$ showing that $\sum_{n=1}^{\infty} u_n$ diverges.

(3) The proof follows from the following example.

Examples 2.1 $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Theorem 2.7 (The D'Alembert's test) Suppose $\sum_{n=1}^{\infty} u_n$ is a series with nonnegative terms. If there is some $N > 0$ such that for all $n > N$,

$$\sqrt[n]{u_n} \leq q < 1,$$

then $\sum_{n=1}^{\infty} u_n$ is convergent; if for any N, there is some $n > N$ such that $\sqrt[n]{u_n} \geq 1$, then $\sum_{n=1}^{\infty} u_n$ is divergent.

Theorem 2.8 (The limit form of D'Alembert's test)

Suppose $\sum_{n=1}^{\infty} u_n$ is a series with nonnegative terms.

(1) If

$$\bar{r} = \overline{\lim_{n\to\infty}} \sqrt[n]{u_n} < 1,$$

then $\sum_{n=1}^{\infty} u_n$ is convergent;

(2) If

$$\underline{r} = \varliminf_{n\to\infty} \sqrt[n]{u_n} > 1,$$

then $\sum_{n=1}^{\infty} u_n$ is divergent;

(3) There is convergent series with nonnegative terms such that $\overline{r} = 1$ and divergent series of nonnegative series for which $\underline{r} = 1$.

The proof is similar to that of the case of Cauchy's test.

Remark 2.9 Theorem 2.8 (3) can be seen from Example 2.1.

Examples 2.2 Discuss the convergence of the following series.

(1) $\sum_{n=1}^{\infty}\frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$; (2) $\sum_{n=1}^{\infty}(\frac{\alpha}{n})^n(\alpha>0)$;

(3) $\sum_{n=1}^{\infty}\frac{4^n}{5^n-3^n}$; (4) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\log\frac{n+1}{n}\right)$.

Solution (1) Since

$$\lim_{n\to\infty}\frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}=\lim_{n\to\infty}\frac{n^n\cdot n^{\frac{1}{n}}}{n^n\cdot\left(1+\frac{1}{n^2}\right)^n}=1\neq 0,$$

it is divergent.

(2) Since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{\alpha}{n}\right)^n}=\lim_{n\to\infty}\frac{\alpha}{n}=0,$$

this series is convergent.

(3) Since

$$\lim_{n\to\infty}\sqrt[n]{\frac{4^n}{5^n-3^n}}=\lim_{n\to\infty}\sqrt[n]{\frac{\left(\frac{4}{5}\right)^n}{1-\left(\frac{3}{5}\right)^n}}=\frac{4}{5}<1,$$

this series is convergent.

(4) It follows from

$$0 \le \frac{1}{n} - \log\frac{n+1}{n} \le \frac{1}{n(n+1)}$$

that $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \log\frac{n+1}{n}\right)$ is convergent.

Examples 2.3 Show $\lim_{n\to\infty}\frac{n!}{n^n} = 0$.

Proof Let $u_n = \frac{n!}{n^n}$. Since

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{1}{\left(1+\frac{1}{n}\right)^n},$$

we know

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{1}{e}<1$$

showing that $\sum_{n=1}^{\infty}u_n$ is convergent. Hence $\lim_{n\to\infty}\frac{n!}{n^n}=0$.

Homework Page 23: 1 (1, 3, 5, 7, 9, 11, 13); 2; 4; 5 (1)