Lecture 30 Upper limits and lower limits

§ 1 The convergence of series and their basic properties

1.1 Definition

1.1.1 The general form of a series

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \dots .$$

where u_n is called the general term of $\sum_{n=0}^{\infty} u_n$.

2.1.2 Partial sums

Let $S_n = \sum_{i=0}^{n} u_i$.

Then S_n is called the n-th partial sum of $\sum_{n=0}^{\infty} u_n$. Also we get a sequence $\{S_n\}$ which is called the sequence of the partial sums of $\sum_{n=0}^{\infty} u_n$.

1.1.3 Convergence of a series

For a series $\sum_{n=0}^{\infty} u_n$, let $S_n = \sum_{i=0}^{n} u_i$. If $\lim_{n \to +\infty} S_n = S$, then we call $\sum_{n=0}^{\infty} u_n$ is convergent, and we denote it by

$$\sum_{n=0}^{\infty} u_n = S.$$

Otherwise, $\sum_{n=0}^{\infty} u_n$ is called divergent.

1.1.4 The rest term

$$r_n = S - S_n = \sum_{n=0}^{\infty} u_i = u_{n+1} + u_{n+2} + \cdots.$$

is called the $n-th$ rest term of $\sum_{n=0}^{\infty} u_n$.

1.2 Properties

Proposition 2.2.1 If $\sum_{n=0}^{\infty} u_n$ is convergent, then for any constant a, $\sum_{n=0}^{\infty} au_n$ is still convergent and $\sum_{n=1}^{\infty} a\, u_n = a \sum_{n=1}^{\infty} u_n$.

The proof easily follows from the definition of convergence.

Proposition 2.2.2 Suppose both series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ converge. Then $\sum_{n=0}^{\infty} (u_n \pm v_n)$ converges and

$$\sum_{n=0}^{\infty} (u_n \pm v_n) = \sum_{n=0}^{\infty} u_n \pm \sum_{n=0}^{\infty} v_n.$$

The proof is easy.

Proposition 2.2.3 Suppose $\sum u_n$ is convergent. Then any new series obtained by inserting parentheses is convergent, and its sum is also $\sum_{n=0}^{\infty} u_n$.

That is,

$$(u_1 + u_2 + \ldots + u_{i1}) + (u_{i1+1} + \ldots + u_{i2}) + \ldots = \sum_{n=0}^{\infty} u_n.$$

Proof Suppose $\{S_n\}$ is the sequence of the partial sums of $\sum_{n=0}^{\infty} u_n$, and $\{A_n\}$ is the sequence of the partial sums of the new series obtained by inserting parentheses in $\sum_{n=0}^{\infty} u_n$.

Then

$$A_1 = u_1 + u_2 + \ldots + u_{i1} = S_1,$$

$$A_2 = (u_1 + u_2 + \ldots + u_{i1}) + (u_{i1+1} + \ldots + u_{i2}) = S_{i2},$$

$$\cdots$$

$$A_n = (u_1 + u_2 + \ldots + u_{i1}) + \ldots + (u_{i_{n-1}+1} + \ldots + u_{i_{n1}}) = S_{in},$$

$$\cdots.$$

This shows that $\{A_n\}$ is a subsequence of $\{S_n\}$. Hence $\{A_n\}$ is convergent and converges to the same limit.

Remark 2.2.1 If a series obtained by inserting parentheses in $\sum_{n=0}^{\infty} u_n$ is convergent, $\sum_{n=0}^{\infty} u_n$ itself need not be convergent.

Example 2.2.1 Suppose $\sum_{n=0}^{\infty} (-1)^n$. Obviously, the new series

$$(1-1)+(1-1)+(1-1)+\cdots$$

is convergent, but $\sum_{n=0}^{\infty} (-1)^n$ is divergence, which easily follows from the following proposition.

Proposition 2.2.4 (A necessary condition) If $\sum_{n=0}^{\infty} u_n$ is convergent, then $u_n \to 0$ as $n \to \infty$.

Proof Let S_n be the nth partial sum of $\sum_{n=0}^{\infty} u_n$. Then $\{S_n\}$ is convergent. It follows from

$$u_n = S_n - S_{n-1}.$$

Hence $\lim_{n\to\infty} u_n = S - S = 0$.

A direct consequence of Proposition 2.2.4 is as follows

Corollary 2.2.5 If u_n does not converge to 0, then $\sum_{n=0}^{\infty} u_n$ is divergent.

Remark 2.2.2 The converse of Proposition 2.2.4 is not always true.

Example 2.2.2 $1+\underbrace{\frac{1}{2}+\frac{1}{2}}_{2}+\underbrace{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}_{3}+\cdots+\underbrace{\frac{1}{n}+\cdots+\frac{1}{n}}_{n}+\frac{1}{n+1}+\cdots.$

Obviously, $u_n \to 0$ as $n \to \infty$, but this series is divergent by Proposition 2.2.3.

Proposition 2.2.6 (The Cauchy convergence criterion for a series) The series $\sum_{n=0}^{\infty} u_n$ converges if and only if for every

$\varepsilon > 0$, there exists some N such that inequalities

$m \geq n > N$

$$|S_m - S_n| = |a_{n+1} + \cdots + a_m| < \varepsilon$$

if and only if for every $\varepsilon > 0$, there exists N such that

for any $n > N$ and $p > 0$,

$$|u_{n+1} + \ldots + u_{n+p}| < \varepsilon .$$

The proof is obvious.

Corollary 2.2.7 A new series obtained by adding or cancelling finitely many terms in a convergent (resp. divergent) series $\sum_{n=0}^{\infty} u_n$ is still convergent

(resp. divergent).

Example 2.2.3 Use the Cauchy convergence criterion to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof For any $p > 0$,

$$|S_{n+p} - S_n| = \frac{1}{(n+1)^2} + \dots + \frac{1}{(n+p)^2}$$

$$\le \left(\frac{1}{n} - \frac{1}{n+1}\right) + \dots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right)$$

$$=\frac{1}{n}-\frac{1}{n+p}<\frac{1}{n}.$$

For any $\varepsilon>0$, let $N=[\frac{1}{\varepsilon}]+1$. Then for any $n>N$ and $p>0$,

$$|S_{n+p}-S_n|<\varepsilon.$$

The proof is finished.

Example 2.2.4 Prove that $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Proof For any $p>0$,

$$|S_{n+p}-S_n|=\frac{1}{n+1}+\ldots+\frac{1}{n+p}$$

$$> \frac{1}{n+p} + \ldots + \frac{1}{n+p} = \frac{p}{n+p}.$$

Let $p = n$. Then

$$|S_{2n} - S_n| > \frac{1}{2}.$$

The Cauchy convergence criterion implies that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Example 2.2.5 Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent.

Proof For any $p > 0$,

$$|S_{n+p} - S_n| = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} + \ldots + (-1)^{p-1} \frac{1}{n+p}.$$

If p is odd, then

$$|S_{n+p}-S_n|=\frac{1}{n+1}-(\frac{1}{n+2}-\frac{1}{n+3})-\dots-(\frac{1}{n+p-1}-\frac{1}{n+p})$$

$$<\frac{1}{n+1}.$$

If p is even, then

$$|S_{n+p}-S_n|=\frac{1}{n+1}-(\frac{1}{n+2}-\frac{1}{n+3})-\dots-(\frac{1}{n+p-2}-\frac{1}{n+p-1})-\frac{1}{n+p}.$$

These imply that $|S_{n+p}-S_n|<\frac{1}{n+1}$. For any $\varepsilon>0$, let

$N=[\frac{1}{\varepsilon}]$.

Then for any $n > N$ and any $p > 0$,

$$|S_{n+p} - S_n| < \varepsilon .$$

The proof is completed.

Homework Page 14: 1(1, 2, 3, 4).

Page 15: 2(2); 3.