Chapter 9

Lecture 29 Upper limits and lower limits

§ 1 Upper limits and lower limits

Suppose $\{a_n\}$ is a bounded sequence. Then for any k, $\{a_{k+1}, \cdots, a_n, \cdots\}$ is still bounded. We use β_k to denote the supremum of $\{a_{k+1}, \ldots, a_n, \ldots\}$ and α_k its infimum.

That is

$$\beta_k = \sup_{n>k} \{a_n\} = \sup\{a_{k+1}, \ldots, a_n, \ldots\}$$

and

$$\alpha_k = \inf_{n>k} \{a_n\} = \inf\{a_{k+1}, \ldots, a_n, \ldots\}.$$

In this way, we obtain two sequences $\{\beta_k\}$ which is decreasing and $\{\alpha_k\}$ which is increasing. Hence both $\lim_{k\to\infty} \alpha_k$ and $\lim_{k\to\infty} \beta_k$ exist which are denoted by

$$H = \overline{\lim_{n\to\infty}} a_n = \lim_{k\to\infty} \sup_{n>k} \{a_n\}$$

and

$$h = \underline{\lim_{n\to\infty}} a_n = \lim_{k\to\infty} \inf \{a_n\}.$$

Proposition 1.1.1 $h \le H$.

Remark 1.1.1 If $\{a_n\}$ has no upper bounded, then we assume $H = \overline{\lim_{n\to\infty}} a_n = +\infty$. If $\{a_n\}$ has no lower bound, then we assume $h = \lim_{n\to\infty} a_n = -\infty$.

Theorem 1.1.2 Suppose $H = \overline{\lim_{n\to\infty}} a_n$.

(1) If H is finite, then for any $\varepsilon > 0$, the interval

$(H-\varepsilon, H+\varepsilon)$ contains infinitely many a_n, but

$(H+\varepsilon, +\infty)$ contains any finitely many a_n;

(2) If $H = +\infty$, then for any $G > 0$, there are infinitely many

a_n such that $a_n > G$.

(3) If $H = -\infty$, then $\lim_{n\to\infty} a_n = H$.

Proof (1) **Claim I** For any $\varepsilon > 0$, there are infinitely many a_n such that

$$a_n > H - \varepsilon .$$

Suppose not. Then there exist $\varepsilon_0 > 0$ and n_0 such that for all $n > n_0$,

$$a_n \le H - \varepsilon_0 .$$

It follows that for all $n > n_0$,

$$\beta_n = \sup\{a_{n+1}, \cdots, a_{n+2}, \cdots\} \le H - \varepsilon_0 ,$$

which implies that

$$H \le H - \varepsilon_0 .$$

This is the desired contradiction.

Claim II For any $\varepsilon > 0$, there are at most finitely a_n such that

$$a_n \geq H + \varepsilon .$$

It follows from $\lim_{n\to\infty} \beta_n = H$ that for any $\varepsilon > 0$, there is N such that for all $n > N$,

$$\beta_n < H + \varepsilon .$$

Since $\beta_n = \sup\{a_{n+1}, \cdots, a_{n+2}, \cdots\}$, we see that for any $k > 0$,

$$a_{n+k} \leq \beta_n < H + \varepsilon .$$

The proof of (1) follows from the combination of Claims I and II.

(2) If $H=+\infty$, then $\{a_n\}$ is unbounded. The conclusion is obvious.

(3) If $H=-\infty$, then for any $G>0$, there is some n_0 such that for all $n>n_0$,

$$a_{n+1}\le\beta_n<-G.$$

This shows that $\lim_{n\to\infty} a_n=-\infty$.

Similar discussion as in the proof of Theorem 1.1.2 shows that

Theorem 1.1.3 Suppose $h = \varliminf_{n\to\infty} a_n$.

(1) If h is finite, then for any $\varepsilon > 0$, the interval $(h-\varepsilon, h+\varepsilon)$ contains infinitely many a_n and $(-\infty, h-\varepsilon)$ contains only finitely many a_n;

(2) If $h=-\infty$, then for any $N>0$, there are infinitely many a_n such that $a_n < -N$;

(3) If $h=+\infty$, then $\lim_{n\to\infty} a_n = +\infty$.

Theorem 1.1.4 Suppose

$$H = \overline{\lim_{n\to\infty}} a_n \quad \text{and} \quad h = \underline{\lim_{n\to\infty}} a_n .$$

Then

$$H = \max\{A:\ A = \lim_{k\to\infty} a_{n_k} \textit{ for any convergent subsequence } \{a_{n_k}\} \textit{ of } \{a_n\}\};$$

$$h = \max\{B:\ B = \lim_{k\to\infty} a_{n_k} \textit{ for any convergent subsequence } \{a_{n_k}\} \textit{ of } \{a_n\}\}.$$

Proof It suffices to prove the first conclusion. The second one follows from similar reasoning. We divided our discussions into three cases.

Case I H is finite

Then Theorem 1.1.2 implies that there must be a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\lim_{n\to\infty} a_{n_k} = H$$

and $(H+\varepsilon, +\infty)$ contains only finitely many $\{a_n\}$. Hence for any convergent subsequence $\{a'_{n_k}\}$,

$$\lim_{k\to\infty} a'_{n_k} \le H+\varepsilon .$$

By the arbitrariness of ε, we see that

$$\lim_{k\to\infty} a'_{n_k} \le H .$$

Case II $H=+\infty$

Then Theorem 1.1.2 implies that there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\lim_{n\to\infty} a_{n_k} = +\infty .$$

Case III $H=-\infty$

Theoem 1.1.2 shows that $\lim_{n\to\infty} a_n = -\infty$.

The conclusion easily follows.

Theorem 1.1.4 implies the following.

Corollary 1.1.5 $\lim_{n\to\infty} a_n = A$ (finite or infinite) if and only $\overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n = A$.

Example 1.1.1 Suppose $a_n = n + (-1)^n n \quad (n = 1, 2, \ldots)$. Find $\overline{\lim_{n\to\infty}}\, a_n$ and $\underline{\lim_{n\to\infty}}\, a_n$.

Solution Since $a_n = n + (-1)^n n = \begin{cases} 2n, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$, we see that

$$\overline{\lim_{n\to\infty}}\, a_n = \infty \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, a_n = 0.$$

Example 1.1.2 Suppose $a_n = n\cos\frac{n}{4}\pi \ (n = 0, 1, 2, \ldots)$. Find $\overline{\lim_{n\to\infty}}\, a_n$ and $\underline{\lim_{n\to\infty}}\, a_n$.

Solution Since

$$a_n = n\cos\frac{n}{4}\pi = \begin{cases} 8k, & n = 8k \\ \frac{8k+1}{2}\sqrt{2}, & n = 8k+1 \\ 0, & n = 8k+2 \\ -\frac{8k+3}{2}, & n = 8k+3 \\ -(8k+4), & n = 8k+4 \\ -\frac{8k+5}{2}, & n = 8k+5 \\ 0, & n = 8k+6 \\ \frac{8k+7}{2}, & n = 8k+7 \end{cases}$$

, we have that

$$\overline{\lim_{n\to\infty}} a_n = +\infty \text{ and } \underline{\lim_{n\to\infty}} a_n = -\infty .$$

Example 1.1.3 Show $\overline{\lim_{n\to\infty}} (a_n + b_n) \le \overline{\lim_{n\to\infty}} a_n + \overline{\lim_{n\to\infty}} b_n$.

Proof The proof easily follows from the following inequality:

$$\sup\{a_n + b_n\} \le \sup\{a_n\} + \sup\{b_n\}.$$

Example 1.1.4 Show $\overline{\lim_{n\to\infty}} (a_n \cdot b_n) \le \overline{\lim_{n\to\infty}} a_n \cdot \overline{\lim_{n\to\infty}} b_n$.

Proof The proof easily follows from the following inequality:

$$\sup\{a_n \cdot b_n\} \le \sup\{a_n\} \cdot \sup\{b_n\}.$$

Example 1.1.5 Suppose $\lim\limits_{n\to\infty} a_n$ exists. Show

$$\overline{\lim_{n\to\infty}}(a_n+b_n)=\lim_{n\to\infty} a_n+\overline{\lim_{n\to\infty}} b_n.$$

Proof **(I)** It follows from **Example** 1.1.3 that

$$\overline{\lim_{n\to\infty}}(a_n+b_n)\le\overline{\lim_{n\to\infty}} a_n+\overline{\lim_{n\to\infty}} b_n=\lim_{n\to\infty} a_n+\overline{\lim_{n\to\infty}} b_n$$

and

$$\overline{\lim_{n\to\infty}} b_n\le\overline{\lim_{n\to\infty}}(a_n+b_n)+\overline{\lim_{n\to\infty}}(-a_n)=\overline{\lim_{n\to\infty}}(a_n+b_n)-\lim_{n\to\infty} a_n.$$

The equality easily follows.

(II) Let $\lim_{n\to\infty} a_n = a$. Then for any $\varepsilon > 0$, there is some $N > 0$ such that for all $n > N$,

$$a - \varepsilon < a_n < a + \varepsilon.$$

It follows that

$$a - \varepsilon + b_n < a_n + b_n < a + \varepsilon + b_n.$$

So $\sup\{a - \varepsilon + b_n\} < \sup\{a_n + b_n\} < \sup\{a + \varepsilon + b_n\}$,

which is

$$a - \varepsilon + \sup\{b_n\} < \sup\{a_n + b_n\} < a + \varepsilon + \sup\{b_n\}.$$

Hence

$$a+\overline{\lim_{n\to\infty}} b_n-\varepsilon \le \overline{\lim_{n\to\infty}}(a_n+b_n)\le a+\varepsilon+\overline{\lim_{n\to\infty}} b_n,$$

which implies the required equality.

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