矢量分析

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2013.8.28

teaching objectives

1. 矢量代数;
2. 梯度、散度和旋度;
3. 两条定理;
4. ∇运算公式;
5. 正交坐标系;
6. 并矢张量.

- 混合积 $\vec{a}\cdot(\vec{b}\times\vec{c})$
- 矢积$\vec{a}\times(\vec{b}\times\vec{c})$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}) \qquad (1)$$

混合积的几何意义：平行六面体的体积。注意$\vec{a},\vec{b},\vec{c}$三者必须满足右手关系，以保证体积为正.

$$\vec{c}\times(\vec{a}\times\vec{b})=\vec{a}(\vec{b}\cdot\vec{c})-\vec{b}(\vec{c}\cdot\vec{a}) \qquad (2)$$

证明如下:

令$\vec{d}=\vec{a}\times\vec{b},\vec{f}=\vec{c}\times\vec{d}$,有$\vec{f}$的$x$分量为:

$$
\begin{aligned}
f_1 &= c_2 d_3 - c_3 d_2 \\
&= c_2(a_1 b_2 - a_2 b_1) - c_3(a_3 b_1 - a_1 b_3) \\
&= a_1(c_2 b_2 + c_3 b_3) - b_1(c_2 a_2 + c_3 a_3) \\
&= a_1(\vec{c} \cdot \vec{b}) - b_1(\vec{c} \cdot \vec{a})
\end{aligned}
$$

同理,可得 f_2, f_3的结果,故

$$
\vec{c} \times (\vec{a} \times \vec{b}) = \vec{a}(\vec{b} \cdot \vec{c}) - \vec{b}(\vec{c} \cdot \vec{a})
$$

$$
(\vec{a} \times \vec{b}) \times \vec{c} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{a}(\vec{b} \cdot \vec{c})
$$

请观察总结一下这两式的规律.

- 梯度:沿变化率最大方向的导数.
- 散度:单位体积的发散量.
- 旋度:单位面积的环量.

引入算符号 ∇, 在直角坐标系中,

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \qquad (3)$$

∇ 算符具有双重性,它既是微分算符,又是矢量.所以运算时既要对函数进行微分,同时又要考虑它的矢量性.

- 梯度

$$
\begin{aligned}
d\vec{\ell} &= dx\,\vec{e}_x + dy\,\vec{e}_y + dz\,\vec{e}_z \\
d_\varphi &= \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz \\
&= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)\varphi \cdot d\vec{\ell} \\
&= \nabla\varphi \cdot d\vec{\ell}
\end{aligned}
$$

$\frac{d\varphi}{d\ell}$ 为 $\vec{\ell}$ 方向的导数，$\frac{d\varphi}{d\ell} = |\nabla\varphi|\cos\theta$，显然 $\nabla\varphi$ 是变化率最大方向的导数–梯度.

$$
\nabla\varphi = \vec{e}_x\frac{\partial\varphi}{\partial x} + \vec{e}_y\frac{\partial\varphi}{\partial y} + \vec{e}_z\frac{\partial\varphi}{\partial z}
$$

- 散度

$$div\vec{f} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{f} \cdot d\vec{S}}{\Delta V} \tag{4}$$

$$\nabla \cdot \vec{f} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{f} \cdot d\vec{S}}{\Delta V}$$

$$\oint_S \vec{f} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{f}) dV$$

$$\nabla \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

- 旋度

$$(rot\vec{A})_n = \lim_{\Delta S \to 0} \frac{\oint_L d\vec{l} \cdot \vec{A}}{\Delta S} \tag{5}$$

$$(\nabla \times \vec{A})_n = \lim_{\Delta S \to 0} \frac{\oint_L d\vec{l} \cdot \vec{A}}{\Delta S} \tag{6}$$

$$\oint_L d\vec{l} \cdot \vec{A} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \int_S (\nabla \times \vec{A})_n dS \tag{7}$$

在直角坐标系下，旋度的表达式为:

$$\nabla \times \vec{f} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} \quad (8)$$

3. 两个定理:

- 梯度的旋度等于零: $\nabla \times \nabla \varphi = 0$
- 旋度的散度等于零: $\nabla \cdot (\nabla \times \vec{A}) = 0$

证明（定理1）：令:

$$\vec{f} = \nabla\varphi$$

$$\begin{aligned}\left(\nabla\times\nabla\varphi\right)_x &= \left(\nabla\times\vec{f}\right)_x \\ &= \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \\ &= \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial y}\right) \\ &= 0\end{aligned}$$

同理可得其它分量，故

$$\nabla\times\nabla\varphi\equiv 0$$

证明（定理2）：

$$\nabla\cdot(\nabla\times\vec{f}) = \frac{\partial}{\partial x}(\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial f_x}{\partial z}-\frac{\partial f_z}{\partial x})$$

$$+\frac{\partial}{\partial z}(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y})$$

$$=0$$

逆定理同样成立，即:

- 无旋场必可表为标量场的梯度.

$$\nabla\times\vec{f}=0, \vec{f}=\nabla\varphi$$

4.∇ 算符的运算公式

- 无源场必可表为另一矢量的旋度.

$$\nabla \cdot \vec{f} = 0, \vec{f} = \nabla \times \vec{A}$$

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi \tag{9}$$

$$\nabla \cdot (\varphi\vec{f}) = (\nabla\varphi) \cdot \vec{f} + \varphi\nabla \cdot \vec{f} \tag{10}$$

$$\nabla \times (\varphi\vec{f}) = (\nabla\varphi) \times \vec{f} + \varphi\nabla \times \vec{f} \tag{11}$$

$$\nabla \cdot (\vec{f} \times \vec{g}) = (\nabla \times \vec{f}) \cdot \vec{g} - \vec{f} \cdot (\nabla \times \vec{g}) \tag{12}$$

$$\nabla \times (\vec{f} \times \vec{g}) = (\vec{g} \cdot \nabla)\vec{f} + (\nabla \cdot \vec{g})\vec{f} - (\vec{f} \cdot \nabla)\vec{g} - (\nabla \cdot \vec{f})\vec{g} \tag{13}$$

4.∇ 算符的运算公式

$$\nabla(\vec{f}\cdot\vec{g}) = \vec{f}\times(\nabla\times\vec{g}) + (\vec{f}\cdot\nabla)\vec{g} + \vec{g}\times(\nabla\times\vec{f}) + (\vec{g}\cdot\nabla)\vec{f} \quad (14)$$

$$\nabla\cdot\nabla\varphi = \nabla^2\varphi \quad (15)$$

$$\nabla\times(\nabla\times\vec{f}) = \nabla(\nabla\cdot\vec{f}) - \nabla^2\vec{f} \quad (16)$$

(1) 直角坐标系：

$$\nabla\varphi = \vec{e}_x\frac{\partial\varphi}{\partial x} + \vec{e}_y\frac{\partial\varphi}{\partial y} + \vec{e}_z\frac{\partial\varphi}{\partial z} \tag{17}$$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{18}$$

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{19}$$

$$\nabla^2\varphi = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{20}$$

(2) 柱坐标系：

$$\nabla\varphi = \vec{e}_r \frac{\partial\varphi}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \vec{e}_z \frac{\partial\varphi}{\partial z} \tag{21}$$

$$\nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \tag{22}$$

$$\nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & \vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} \tag{23}$$

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{24}$$

(3) 球坐标系：

$$\nabla\varphi = \vec{e}_r\frac{\partial\varphi}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi} \tag{25}$$

$$\nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{26}$$

$$\nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{27}$$

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\varphi}{\partial\phi^2} \tag{28}$$

- 并矢与张量

并矢：两个矢量放在一起就是并矢.如:

$$\vec{A}\vec{B} = \sum_{i,j=1}^{3} A_i B_j \hat{e}_i \hat{e}_j$$

$\vec{A}\vec{B} \neq \vec{B}\vec{A}$,张量是具有九个分量的物理量.如张量T 的九个分量分别为:

$$\begin{matrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{matrix}$$

$$\mathrm{T} = \sum_{i,j=1}^{3} T_{ij}\hat{e}_i\hat{e}_j$$

单位张量:$\mathrm{g} = \sum_{i=1}^{3} \hat{e}_i\hat{e}_i$

张量的运算

$$\vec{A}\vec{B}\cdot\vec{C}=\vec{A}(\vec{B}\cdot\vec{C})=\vec{A}(\vec{C}\cdot\vec{B})=\vec{A}\vec{C}\cdot\vec{B}$$
$$=(\vec{C}\cdot\vec{B})\vec{A}=\vec{C}\cdot\vec{B}\vec{A}$$
$$=(\vec{B}\cdot\vec{C})\vec{A}=\vec{B}\cdot\vec{C}\vec{A}$$

$$\vec{A}\vec{B}\times\vec{C}=\vec{A}(\vec{B}\times\vec{C})$$
$$\vec{C}\times\vec{A}\vec{B}=(\vec{C}\times\vec{A})\vec{B}$$
$$\vec{A}\vec{B}:\vec{C}\vec{D}=(\vec{B}\cdot\vec{C})(\vec{A}\cdot\vec{D})$$
$$\mathrm{g}\cdot\vec{A}\vec{B}=\vec{A}\vec{B}\cdot\mathrm{g}=\vec{A}\vec{B}$$

张量分析

$$\nabla \cdot (\vec{f}\vec{g}) = (\nabla \cdot \vec{f})\vec{g} + (\vec{f} \cdot \nabla)\vec{g}$$

$$\nabla \cdot \mathrm{T} = \frac{\partial}{\partial x}(\vec{e}_1 \cdot \mathrm{T}) + \frac{\partial}{\partial y}(\vec{e}_2 \cdot \mathrm{T}) + \frac{\partial}{\partial z}(\vec{e}_3 \cdot \mathrm{T})$$

$$\oint d\vec{S} \cdot \mathrm{T} = \int dV \nabla \cdot \mathrm{T}$$

$$\oint d\vec{S} \cdot (\vec{f}\vec{g}) = \int dV \nabla \cdot (\vec{f}\vec{g})$$